# On Fractional Boundary Value Problems with Positive and Increasing Solutions

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Let $J=[0,1]$ and $\mathbb{R}_0=[0,\infty)$.

We consider the fractional boundary value problem

$$ {}^c\!D^\alpha u(t)=q\big(t,u(t),u'(t)\big){}^c\!D^\beta u(t)+f\big(t,u(t),u'(t)\big), \tag{1}$$

$$ u(0)=ku'(0), \quad u(1)=ku'(1), \;\; k\geq\frac{1}{\alpha-1}\,, \tag{2}$$

where $1<\beta<\alpha\leq 2$, ${}^c\!D$ denotes the Caputo fractional derivative and

$(H_1)$ $f,q\in C(J\times\mathbb{R}_0^2)$ and

$$0\leq f(t,x,y), \;\; 0\leq q(t,x,y)\leq W<\infty \;\text{ for }\; (t,x,y)\in J\times\mathbb{R}_0^2. \tag{3}$$

The further conditions on $f$ will be specified later.

We recall that *the Riemann–Liouville fractional integral* $I^\gamma x$ of order $\gamma>0$ of a function $x:J\to\mathbb{R}$ is defined as [1,2]

$$I^\gamma x(t)=\int\limits_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\,x(s)\,\mathrm{d}s$$

and *the Caputo fractional derivative* ${}^c\!D^\gamma x$ of order $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x:J\to\mathbb{R}$ is given as

$$ {}^c\!D^\gamma x(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_0^t\frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)}\Big(x(s)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}\,s^k\Big)\,\mathrm{d}s,$$

provided that the right-hand sides exist. Here, $\Gamma$ is the Euler gamma function and $n=[\gamma]+1$, $[\gamma]$ means the integral part of the fractional number $\gamma$. $\Lambda^0$ is the identical operator and if $n\in\mathbb{N}$, then ${}^c\!D^n x(t)=x^{(n)}(t)$.

In particular,

$$ {}^c\!D^\gamma x(t)=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int\limits_0^t\frac{(t-s)^{1-\gamma}}{\Gamma(2-\gamma)}\,\big(x(s)-x(0)-x'(0)s\big)\,\mathrm{d}s, \;\; \gamma\in(1,2).$$

**Definition.** We say that $u$ is *a solution of equation* (1) if $u\in C^1(J)$, ${}^c\!D^\alpha u\in C(J)$ and (1) holds for $t\in J$. A solution $u$ of (1) satisfying the boundary condition (2) is called *a solution of problem* (1), (2). We say that $u$ is *a positive and increasing solution of problem* (1), (2) if $u>0$ and $u'>0$ on $J$.

The special case of problem (1), (2) is the problem

$$u''(t) = q\big(t, u(t), u'(t)\big){}^{c}\!D^{\beta} u(t) + f\big(t, u(t), u'(t)\big), \tag{4}$$

$$u(0) = ku'(0), \quad u(1) = ku'(1), \;\; k \geq 1. \tag{5}$$

Equation (4) is called the generalized Bagley–Torvik fractional differential equation (see [2–6]).

We are interested in the existence of positive and increasing solutions to problem (1), (2). To this end for $a \in C(J)$ introduce an operator $\Lambda_a : C(J) \to C(J)$ as

$$\Lambda_a x(t) = a(t) I^{\alpha-\beta} x(t).$$

For $n \in \mathbb{N}$, let $\Lambda_a^n = \underbrace{\Lambda_a \circ \Lambda_a \circ \cdots \circ \Lambda_a}_{n}$ be $n$th iteration of $\Lambda_a$ and $\mathcal{B}_a$ be an operator acting on $C(J)$ defined by the formula

$$\mathcal{B}_a x(t) = \sum_{n=0}^{\infty} \Lambda_a^n x(t).$$

For $\gamma > 0$, let $E_\gamma$ be the classical Mittag–Leffler functions [1, 2]

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma + 1)}\,, \quad z \in \mathbb{R}.$$

In the following result, solutions of the auxiliary linear fractional differential equation

$${}^{c}\!D^{\alpha} u(t) = a(t){}^{c}\!D^{\beta} u(t) + r(t), \tag{6}$$

satisfying (2), are given by the operator $\mathcal{B}_a$.

**Lemma 1.** *Let $a, r \in C(J)$. Then the function*

$$u(t) = I^{\alpha} \mathcal{B}_a r(t) + (t + k)\Big(kI^{\alpha-1} \mathcal{B}_a r(t)\big|_{t=1} - I^{\alpha} \mathcal{B}_a r(t)\big|_{t=1}\Big), \;\; t \in J,$$

*is the unique solution to problem* (6), (2)*.*

Let

$$\mathcal{S} = \big\{x \in C^1(J) : \;\; x(t) \geq 0, \;\; x'(t) \geq 0 \;\text{ for }\; t \in J\big\}$$

and, under condition $(H_1)$, introduce the Nemytskii operators $\mathcal{Q}, \mathcal{F} : \mathcal{S} \to C(J)$,

$$\mathcal{Q}x(t) = q\big(t, x(t), x'(t)\big), \quad \mathcal{F}x(t) = f\big(t, x(t), x'(t)\big),$$

where $q$ and $f$ are from (1). It is clear that $\mathcal{S}$ is a cone in $C^1(J)$. Note that, by the definition,

$$\Lambda_{\mathcal{Q}x} y(t) = q\big(t, x(t), x'(t)\big) I^{\alpha-\beta} y(t).$$

Keeping in mind, Lemma 1 define an operator $\mathcal{K}$ acting on $\mathcal{S}$ by the formula

$$\mathcal{K}x(t) = I^{\alpha} \mathcal{L}_{\mathcal{Q}x} x(t) + (t + k)\Big(kI^{\alpha-1} \mathcal{L}_{\mathcal{Q}x} x(t)\big|_{t=1} - I^{\alpha} \mathcal{L}_{\mathcal{Q}x} x(t)\big|_{t=1}\Big),$$

where

$$\mathcal{L}_{\mathcal{Q}x} x(t) = \mathcal{B}_{\mathcal{Q}x} \mathcal{F}x(t)$$

and $k \geq 1/(\alpha - 1)$ is from (2).

The properties of $\mathcal{K}$ are summarized in the following lemma.

**Lemma 2.** *Let* $(H_1)$ *hold. Then* $\mathcal{K}:\mathcal{S}\to\mathcal{S}$, $\mathcal{K}$ *is a completely continuous operator and if* $u$ *is a fixed point of* $\mathcal{K}$, *then* $u$ *is a solution to problem* (1), (2).

In view of Lemma 2, we need to prove that the operator $\mathcal{K}$ admits a fixed point. The existene of a fixed point of $\mathcal{K}$ is proved in Theorem 1 by the Schauder fixed point theorem, while in Theorem 2 by the Guo–Krasnoselskii fixed point theorem on cones. We work with the following growth condition on the function $f$.

$(H_2)$ For $t\in J$ and $x,y\in\mathbb{R}_0$, the estimate

$$f(t,x,y)\leq\varphi(x+y)$$

holds, where $\varphi\in C(\mathbb{R}_0)$, $\varphi$ is positive, nondecreasing and there exists $M>0$ such that

$$\varphi(M)\leq\frac{M\Gamma(\alpha+1)}{(1+k)(\alpha k+\alpha-1)E_{\alpha-\beta}(W)}\,, \tag{7}$$

where $W$ is from $(H_1)$.

**Theorem 1.** *Let* $(H_1)$ *and* $(H_2)$ *hold. Let* $f(t_0,0,0)>0$ *for some* $t_0\in J$. *Then there exists at least one positive and increasing solution to problem* (1), (2).

If $f(t,0,0)=0$ on $J$, we can't apply Theorem 1 to problem (1), (2). In this case $u=0$ is a solution of this problem.

**Example 1.** Let $\rho,\mu\in(0,1)$, $a,p\in C(J)$ and $p(t_0)\neq0$ for some $t_0\in J$. Theorem 1 guarantees that the equation

$${}^cD^\alpha u=\big|a(t)+\cos(x-y)\big|{}^cD^\beta u+|p(t)|+u^\rho+(u')^\mu$$

has at least one positive and increasing solution satisfying condition (2).

**Corollary 1.** *Let* $(H_1)$ *and* $(H_2)$ *with* (7) *replaced by*

$$\varphi(M)\leq\frac{2M}{(1+k)(2k+1)E_{2-\beta}(W)}$$

*hold. Let* $f(t_0,0,0)>0$ *for some* $t_0\in J$. *Then there exists at least one positive and increasing solution to problem* (4), (5).

**Theorem 2.** *Let* $(H_1)$ *and* $(H_2)$ *hold. Let*

$$\lim_{x,y\in\mathbb{R}_0,\,x+y\to0}\frac{f(t,x,y)}{x+y}>\frac{\Gamma(\alpha+1}{2(k\alpha-1)}\ \ \textit{uniformly on } J.$$

*Then problem* (1), (2) *has at least one positive and increasing solution.*

**Example 2.** Let $a,p\in C(J)$ and $p>\frac{\Gamma(\alpha+1)}{2(k\alpha-1)}$. Theorem 2 guarenrees that there exists a positive and increasing solution of the equation

$${}^cD^\alpha u=\big|a(t)+e^{-u}\sin u'\big|{}^cD^\beta u+p(t)(u+u')e^{-u-u'}, \tag{8}$$

satisfying condition (2). Note that $u=0$ is also a solution to problem (8), (2).

## References

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