On Existence of Solutions with Prescribed Number of Zeros to Third Order Emden–Fowler Equations with Singular Nonlinearity and Variable Coefficient

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1 Introduction

The problem of the existence of solutions to Emden–Fowler type equations with prescribed number of zeros on a given domain is investigated.

Consider the equation

$$y''' = -p(t, y, y', y'')|y|_+^k, \quad \text{where } k \in (0,1), \quad 0 < m \leqslant p(t, y_0, y_1, y_2) \leqslant M < \infty, \tag{1.1}$$

function $p(t, y_1, y_2, y_3)$ is continuous and it is Lipschitz continuous in (y_1, y_2, y_3). By $|y|_+^k$ we denote $|y|^k \operatorname{sgn} y$.

The equations similar to (1.1) were considered in the previous papers. The existence of solutions with a given number of zeros on the prescribed interval was proved. In [4] equations of the third- and the fourth- order with constant coefficient and $k \in (0,1) \cup (1,+\infty)$ was investigated. In [6] we provide our results regarding high-order Emden–Fowler type equation with constant coefficient and regular nonlinearity ($k > 1$). This result was proved using a theorem obtained by I. Astashova in [2]. The work [7] contains theorems regarding equation (1.1) with $k > 1$. Now we generalize the result obtained in [7] to the case of singular nonlinearity $k \in (0,1)$.

2 Main result

Theorem 2.1. *For any $k \in (0,1)$, $-\infty < a < b < +\infty$, and integer $j \geqslant 2$, equation (1.1) has a solution defined on the segment $[a,b]$, vanishing at its endpoints, and having exactly j zeros on $[a,b]$.*

The idea of the proof is as follows. In [1] it was proved that any solution $y(t)$ is oscillatory if the conditions $y(a) = 0$, $y'(a) > 0$, $y''(a) > 0$ hold. We cannot rely on the Continuous Dependence On Parameters theorem, because its conditions do not fulfill. Nevertheless, solutions to (1.1) (in some extent) are continuous, and we prove this fact. After that we prove that the location of the N-th zero of solution $y(t)$ depends continuously on its initial data. Then we can make upper and lower estimates of that location. Finally, we prove that there exist initial data such that the N-th zero of the related solution $y(t)$ is exactly at the point b.

2.1 Continuous Dependence of Solutions on Initial Data

Lemma 2.1. *Let $y(t)$ be a solution to equation (1.1) defined on $[t_0, I^*]$ and satisfying $y(t_0) = y_0$, $y'(t_0) = y_1 \neq 0$, $y''(t_0) = y_2$. Then there exists $I \in (t_0, I^*)$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any (z_0, z_1, z_2) belonging to the δ-neighborhood of (y_0, y_1, y_2) and any continuous* *in* (t, x_0, x_1, x_2) *and Lipschitz continuous in* (x_0, x_1, x_2) *function* $q(t, x_0, x_1, x_2)$ *satisfying for all* (t, ξ_1, ξ_2, ξ_3) *the inequality*

$$\big|p(t, \xi_1, \xi_2, \xi_3) - q(t, \xi_1, \xi_2, \xi_3)\big| < \delta,$$

the solution $z(t)$ *to the Cauchy problem*

$$\begin{cases} z''' = -q(t, z, z', z'')|z|_+^k, \\ z(t_0) = z_0, \\ z'(t_0) = z_1, \\ z''(t_0) = z_2 \end{cases} \tag{2.1}$$

is extensible onto $[t_0, I]$ *and satisfies on this segment the inequalities*

$$|y(t) - z(t)| < \varepsilon, \quad |y'(t) - z'(t)| < \varepsilon, \quad |y''(t) - z''(t)| < \varepsilon.$$

By integrating equation (1.1) three times and taking into account the initial data, we can obtain that the solution $y(t)$ satisfies

$$y(t) = y_0 + y_1(t - t_0) + y_2\,\frac{(t - t_0)^2}{2} - \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} p(\xi, y, y', y'')|y|_+^k \, d\xi \, d\tau \, d\eta.$$

From this we can obtain the following estimate:

$$\begin{aligned} |z(t) - y(t)| \leqslant |y_0 - z_0| + |z_1 - y_1|\,|t - t_0| + |z_2 - y_2|\,\frac{|t - t_0|^2}{2} \\ \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} \big|(p(\xi, y, y', y'') - q(\xi, z, z', z''))|y|_+^k\big| \, d\xi \, d\tau \, d\eta \\ + \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} |q(\xi, z, z', z'')|\,\big||y|_+^k - |z|_+^k\big| \, d\xi \, d\tau \, d\eta. \end{aligned} \tag{2.2}$$

Our goal is to prove that if the difference of the initial data is small, then the difference of the solutions is small too. For example, take a look at the last term of (2.2)

$$\begin{aligned} \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} |q(\xi, z, z', z'')|\,\big||y|_+^k - |z|_+^k\big| \, d\xi \, d\tau \, d\eta \leqslant M \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} \big||y|_+^k - |z|_+^k\big| \, d\xi \, d\tau \, d\eta \\ = M \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} |y|^k \Big|1 - \Big|\frac{z}{y}\Big|_+^k\Big| \, d\xi \, d\tau \, d\eta \leqslant M \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} |y|^k \,\frac{1}{k}\, |1 - \frac{z}{y}\,| \, d\xi \, d\tau \, d\eta \\ = \frac{M}{k} \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} |y|^{k-1}|y - z| \, d\xi \, d\tau \, d\eta \leqslant \frac{M}{k} \max_{[t_0, I]} |y - z| \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} |y|^{k-1}\,\frac{y'}{y'} \, d\xi \, d\tau \, d\eta \\ \leqslant \max_{[t_0, I]} |y - z|\,\frac{M}{k} \max_{[t_0, I]} \Big|\frac{1}{y'}\Big| \int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\int\limits_{t_0}^{\eta} |y|^{k-1} y' \, d\xi \, d\tau \, d\eta \end{aligned}$$

$$\leqslant \max_{[t_0,I]}|y-z|\,\frac{M}{k}\,\max_{[t_0,I]}\Big|\frac{1}{y'}\Big|\int\limits_{t_0}^{t}\int\limits_{t_0}^{\tau}\frac{1}{k}\,\big||y(I)|^k-|y(t_0)|^k\big|\,d\tau\,d\eta$$
$$= \max_{[t_0,I]}|y-z|L_3(t-t_0)^2\big||y(I)|^k-|y(t_0)|^k\big|. \quad (2.3)$$

Here L_3 depends only on k, $q(t,y_0,y_1,y_2)$ and $y(t)$. From (2.2) we can obtain the following inequality:

$$|y-z|\leqslant L_1\max\big\{|z_0-y_0|,|z_1-y_1|,|z_2-y_2|\big\}$$
$$+L_2(t-t_0)^3\Big(\max_{[t_0,I]}|y-z|+\max_{[t_0,I]}|y'-z'|+\max_{[t_0,I]}|y''-z''|+\max|p-q|\Big)$$
$$+\Big(L_3(t-t_0)^2\big||y(I)|^k-|y(t_0)|^k\big|\Big)\max_{[t_0,I]}|y-z|. \quad (2.4)$$

Similarly, we can integrate equations twice or once and obtain estimates for $|y'-z'|$ and $|y''-z''|$, respectively. Adding all the estimates obtained together, we get the evaluation

$$\Big(\max_{[t_0,I]}|y-z|+\max_{[t_0,I]}|y'-z'|+\max_{[t_0,I]}|y''-z''|\Big)$$
$$\leqslant\frac{K_1}{1-K_2[(I-t_0)]}\Big(\max\{|z_0-y_0|,|z_1-y_1|,|z_2-y_2|,\max|p-q|\}\Big),$$

where $K_1>0$, K_1 and $K_2[x]$ do not depend on ε, and $K_2[x]>0$ is a function tending to zero as $x\to 0$. It is possible to choose I such that $1-K_2[(I-t_0)]>0$.

The evaluation shows that if $\max\{|z_0-y_0|,|z_1-y_1|,|z_2-y_2|,\max|p-q|\}$ is sufficiently small, then

$$\max_{[t_0,I]}|y-z|+\max_{[t_0,I]}|y'-z'|+\max_{[t_0,I]}|y''-z''|<\varepsilon.$$

This proves the theorem.

Theorem 2.2. *Let $y(t)$ be a solution to* (1.1) *with initial data $y(t_0)=y_0\geqslant 0$, $y'(t_0)=y_1>0$, $y''(t_0)=y_2\geqslant 0$. Suppose $y(t)$ is defined on a segment $[t_0,I]$ and has a finite number of zeros on it. Then for every $\varepsilon>0$ there exists $\delta>0$ such that if $z(t)$ is a solution to* (1.1) *with initial data $z(t_0)=z_0$, $z'(t_0)=z_1$, $z''(t_0)=z_2$, and (z_0,z_1,z_2) belongs to the δ-neighborhood of (y_0,y_1,y_2), then $z(t)$ is extensible onto $[t_0,I]$ and satisfies on it the inequalities $|y(t)-z(t)|<\varepsilon$, $|y'(t)-z'(t)|<\varepsilon$, $|y''(t)-z''(t)|<\varepsilon$.*

Using Lemma 2.1, we put segments of fixed length on every zero of $y(t)$. In such segments continuous dependency on initial data is proven by Lemma 2.1. Between those segments either $y(t)>a>0$ or $y(t)<b<0$, and therefore the Continuous Dependence On Parameters theorem holds (because the right-hand side of (1.1) is not Lipschitz continuous only near $y=0$). Combining all the segments, we prove Theorem 2.2.

2.2 Continuous Dependence of Zeros on Initial Data

Theorem 2.3 (see [7])**.** *Let $y(t)$ be a solution to* (1.1) *with initial data $y(t_0)=0$, $y'(t_0)=y_1>0$, $y''(t_0)=y_2>0$. We denote by $T(y_1,y_2)$ the location of the first zero of $y(t)$ after t_0. Then $T(y_1,y_2)$ is a continuous function.*

Theorem 2.4. *Let $y(t)$ be a solution to* (1.1) *with initial data $y(t_0)=0$, $y'(t_0)=y_1>0$, $y''(t_0)=y_2>0$. We denote by $t_n(y_1,y_2)$ the location of the n-th zero of $y(t)$ after t_0. Then $t_n(y_1,y_2)$ is a continuous function, and $|t_n(y_1,y_2)-t_0|$ runs over all positive numbers.*

Now we can prove the main theorem. If we want a solution $y(t)$ to have exactly j zeros on the segment $[a,b]$, we can find suitable initial data for this. Let $y(a)=0$, $y'(a)=c_1>0$, $y'(a)=c_2>0$. Denote by $t_j(c_1,c_2)$ the location of the $(j-1)$-th zero of $y(t)$ after a. It follows from Theorem 2.4 that $|t_j(c_1,c_2)-a|$ is a continuous function, and this function runs over all positive numbers. Therefore, $|t_j(c_1,c_2)-a|=b$ has a solution (c_1^*,c_2^*). If $y(a)=0$, $y'(a)=c_1^*>0$, $y'(a)=c_2^*>0$, then $y(t)$ has exactly j zeros on $[a,b]$, and this proves the theorem.

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