Existence and Asymptotic Properties of Kneser Solutions to Singular Differential Problems

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1 Formulation of the problem

Analytical results presented here are based on a common research with Jana Burkotová and they are contained in the paper [1] where in addition numerical simulations are discussed. In particular, here we study the existence and asymptotic behaviour of Kneser solutions to the nonlinear second order ODE,

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \ \ t \in [0, \infty), \tag{1}$$

satisfying

$$u(0) = u_0 \in (0, L), \ \ 0 \le u(t) \le L \ \text{ for } \ t \in [0, \infty), \tag{2}$$

or

$$u(0) = u_0 \in (L_0, 0), \ \ L_0 \le u(t) \le 0 \ \text{ for } \ t \in [0, \infty), \tag{3}$$

where the interval $[L_0, L]$ is specified in the following way:

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0.$$

Note that equation (1) is singular because we assume that $p(0) = 0$ (see (6)), and therefore there is a time singularity at $t = 0$.

A function u is called *a solution to equation* (1) *on* $[0, \infty)$ if $u \in C^1[0, \infty)$, $pu' \in C^1[0, \infty)$, and u satisfies equation (1) for all $t \in [0, \infty)$. The solution u to equation (1) on $[0, \infty)$ is called *a solution to problem* (1), (2) or problem (1), (3) if u additionally satisfies condition (2) or (3), respectively. A solution u to equation (1) on $[0, \infty)$ is called a *Kneser solution* if there exists $t_0 > 0$ such that

$$u(t)u'(t) < 0 \ \text{ for } \ t \in [t_0, \infty). \tag{4}$$

2 Existence of Kneser solutions to singular equation (1)

In this section, the existence of Kneser solutions to problems (1), (2) and (1), (3) is discussed under the assumptions that f is continuous on $[L_0, L]$, p is continuous on $[0, \infty)$ and $p \equiv q$. For more details see [1] and [5]. For the existence of other types of solutions and a deeper study of this problems see also [2], [3], [4].

Theorem 1. *Let us assume that*

$$f \in \mathrm{Lip}_{loc}(0, L], \quad f(x) > 0 \ \textit{ for } \ x \in (0, L), \tag{5}$$

$$p \in C^1(0, \infty), \ \ p(0) = 0, \ \ p' > 0 \ \textit{ on } \ (0, \infty), \quad \lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0, \tag{6}$$

$$\frac{p'(t)\int\limits_0^t p(s)\,\mathrm{d}s}{p^2(t)} \geq c, \quad t \in (0,\infty), \tag{7}$$

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}\,, \quad x \in (0, A_0], \tag{8}$$

hold for some $c > \frac{1}{2}$ *and* $A_0 \in (0, L)$*, where* $F(x) = \int\limits_0^x f(z)\,\mathrm{d}z$*.*

Then, for each $u_0 \in (0, A_0]$ *there exists a unique Kneser solution* u *to problem* (1), (2) *with* $p \equiv q$*. This solution has the following properties:*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) < 0, \quad t \in (0,\infty).$$

A dual statement for an initial condition u_0 from a negative neighbourhood of zero is given in the following theorem.

Theorem 2. *Let us assume that* (6) *and* (7) *with a constant* $c > \frac{1}{2}$ *hold, and let*

$$f \in \mathrm{Lip}_{loc}[L_0, 0), \quad f(x) < 0 \ \ \textit{for} \ \ x \in (L_0, 0). \tag{9}$$

Further, assume that there exists $B_0 \in (L_0, 0)$ *such that the inequality*

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}\,, \quad x \in [B_0, 0), \tag{10}$$

is satisfied.

Then, for each $u_0 \in [B_0, 0)$*, there exists a unique Kneser solution* u *to problem* (1), (3) *with* $p \equiv q$*. This solution has the following properties:*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) > 0, \quad t \in (0,\infty).$$

To our knowledge, the existence of Kneser solutions to singular problems (1), (2) and (1), (3) with $p(0) = 0$ and $p \neq q$ remains an open problem. Let us note, that the condition $u'(0) = 0$ is necessary for the smoothness of the solution in the case where $p \equiv q$ is an increasing function. To see this, let us consider a solution u to (1), (2) or (1), (3). Since $u \in C^1[0,\infty)$, the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. Since f is continuous on $[L_0, L]$ and $u(0) \in (L_0, L)$, there exist $M > 0$ and $\delta > 0$ such that $|f(u(t))| \leq M$ for $t \in (0, \delta)$. We now integrate (1) and use the monotonicity of p to obtain

$$|u'(t)| = \left|\frac{1}{p(t)}\int\limits_0^t p(s)f(u(s))\,\mathrm{d}s\right| \leq \frac{M}{p(t)}\int\limits_0^t p(s)\,\mathrm{d}s \leq Mt, \quad t \in (0, \delta).$$

Consequently, $u'(0) = 0$ holds.

1 Asymptotic properties of Kneser solutions

This section focuses on properties of Kneser solutions to problems (1), (2) and (1), (3) in the neighbourhood of infinity. Asymptotic formulas for the solutions and for their first derivatives are provided. In the following analysis, we assume that the data functions p and q are regularly varying at infinity and

$$f \in C[L_0, L], \quad xf(x) > 0 \ \text{ for } \ x \in (L_0, 0) \cup (0, L). \tag{11}$$

A function g, which is positive and measurable on $[\tau_0,\infty)$, $\tau_0>0$, is called *regularly varying of index* $\alpha\in\mathbb{R}$ if for each $\lambda>0$

$$\lim_{t\to\infty}\frac{g(\lambda t)}{g(t)}=\lambda^{\alpha}.$$

The set of all regularly varying functions of index α is denoted by $RV(\alpha)$.

Our proofs are based on

Karamata Integration Theorem. *Let $L(t)\in SV$, $c>0$.*

(i) *If $\alpha>-1$, then*

$$\int\limits_c^t s^{\alpha}L(s)\,ds\sim\frac{1}{\alpha+1}\,t^{\alpha+1}L(t)\ \ as\ \ t\to\infty.$$

(ii) *If $\alpha<-1$, then*

$$\int\limits_t^{\infty} s^{\alpha}L(s)\,ds\sim-\frac{1}{\alpha+1}t^{\alpha+1}L(t)\ \ as\ \ t\to\infty.$$

(iii) *If $\alpha=-1$, then*

$$l(t)=\int\limits_c^t\frac{L(s)}{s}\,ds\in SV\ \ and\ \ \lim_{t\to\infty}\frac{L(t)}{l(t)}=0.$$

Note, that if

$$p\in C[0,\infty),\ \ p>0\ \text{ on }\ (0,\infty),\ \ p(0)=0, \tag{12}$$

$$q\in C[0,\infty),\ \ q>0\ \text{ on }\ (0,\infty), \tag{13}$$

then problems $(1),(2)$ and $(1),(3)$ have no Kneser solutions in case that

$$\int\limits_1^{\infty}\frac{\mathrm{d}s}{p(s)}=\infty. \tag{14}$$

This follows from (12), (13), (11) and the following arguments: Let u be a solution to $(1),(2)$. Then, pu' is decreasing for $t>0$. Assume that $pu'\le 0$ for $t\ge t_1>0$. By integrating inequality $p(t)u'(t)<p(t_1)u'(t_1)=K<0$, we obtain

$$u(t)\le u(t_1)+K\int\limits_{t_1}^{t}\frac{\mathrm{d}s}{p(s)}\,.$$

Therefore, as t tends to infinity, $\lim\limits_{t\to\infty}u(t)\le-\infty$ contradicting (2). This means that $u'>0$ on $[t_0,\infty)$. Hence, any solution of $(1),(2)$ is increasing and there exists no Kneser solution to $(1),(2)$. Similar arguments can be given for problem $(1),(3)$. According to the Karamata Integration Theorem, condition (14) is satisfied when $p\in RV(\alpha)$ with $\alpha<1$. For $\alpha=1$, the integral may be convergent (or may not) and hence Kneser solutions to the problem could exist. Therefore, in the following asymptotic analysis, we restrict our attention to the case $\alpha\ge 1$. We first formulate the asymptotic properties of Kneser solutions to problem $(1),(2)$, or $(1),(3)$.

Theorem 3. *Assume that* (11) *holds and that* $p \in RV(\alpha) \cap C[0,\infty)$, $q \in RV(\beta) \cap C[0,\infty)$, $\alpha \geq 1$, $\beta > 0$, $\beta - \alpha > -1$. *Let* u *be a Kneser solution to problem* (1), (2) *or* (1), (3). *Then,*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{15}$$

We finally focus our attention to the first derivatives of Kneser solutions.

Theorem 4. *Assume that* (11) *holds and that* $p \in RV(\alpha) \cap C[0,\infty)$, $\alpha \geq 1$, $q \in RV(\beta) \cap C[0,\infty)$, $\beta > 0$, $\beta - \alpha > -1$, *and in addition*

$$\exists\, r > 1 : \liminf_{x\to 0} \frac{|f(x)|}{|x|^r} > 0, \quad \limsup_{x\to 0} \frac{|f(x)|}{|x|^r} < \infty. \tag{16}$$

Let u *be a Kneser solution to problem* (1), (2) *or* (1), (3). *Then, for any* $\varepsilon > 0$

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+2}{r-1}-\varepsilon} |u(t)| = 0. \tag{17}$$

References

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