An m-Dimensional Linear Pfaff Equation with Arbitrary Characteristic Sets

A. S. Platonov

University of Civil Protection, Ministry of Emergency Situations, Minsk, Belarus
E-mail: `alexpltn@mail.ru`

S. G. Krasovskii

Institute of Mathematics, National Academy of Sciences, Minsk, Belarus
E-mail: `kras@im.bas-net.by`

Consider the linear Pfaff system

$$\frac{\partial x}{\partial t_i} = A_i(t)x, \ \ x \in R^n, \ \ t = (t_1, t_2, \dots, t_m) \in R^m_+, \ \ i = \overline{1, m}, \tag{1}$$

with bounded coefficient matrices $A_i(t)$ continuously differentiable in $R^m_+ = \{t \in R^m : \ t \geq 0\}$ and satisfying the condition of complete integrability [1, pp. 14–24], [2, pp. 16–26]. The characteristic vector [1, p. 83], [3], $\lambda[x] = \lambda$ and the lower characteristic vector [4] $p[x] = p$ of a nontrivial solution $x : R^m_+ \to R^n \setminus \{0\}$ of system (1) is defined by the conditions

$$L_x(\lambda) \equiv \varlimsup_{t\to\infty} \frac{\ln \|x(t)\| - (\lambda, t)}{\|t\|} = 0, \ \ L_x(\lambda - \varepsilon e_i) > 0, \ \ \forall\, \varepsilon > 0, \ \ i = 1, \dots, m, \tag{2}$$

$$l_x(p) \equiv \varliminf_{t\to\infty} \frac{\ln \|x(t)\| - (p, t)}{\|t\|} = 0, \ \ l_x(p + \varepsilon e_i) < 0, \ \ \forall\, \varepsilon > 0, \ \ i = 1, \dots, m, \tag{3}$$

where $e_i = (\underbrace{0, \dots, 0, 1}_{i}, 0, \dots, 0) \in R^m_+$ is a unit coordinate vector. The characteristic set Λ_x [3] and the lower characteristic set P_x [4] of a nontrivial solution $x : R^m_+ \to R^n \setminus \{0\}$ of system (1) is defined as the unions of all characteristic vectors $\Lambda_x = \cup\lambda[x]$ and all lower characteristic vectors $P_x = \cup p[x]$ of that solution. The sets [3], [4] $\Lambda(A) = \bigcup\limits_{x\neq 0} \Lambda_x$ and $P(A) = \bigcup\limits_{x\neq 0} P_x$ referred respectively to as the characteristic and the lower characteristic sets of system (1).

We generalize the statement on joint implementation of the characteristic and the lower characteristic sets of the linear Pfaff system (1) with two-dimensional time ($m = 2$) [6] on the system (1) with m-dimensional time t.

Definition 1 ([9])**.** A set $D \subset R^m$ is said to be *bounded above* (respectively, *below*) if there exists an $r \in R^m$ such that $d \leq r$ (respectively, $d \geq r$) for all $d \in D$ ($d \leq r$ is equivalent to the inequalities $d_i \leq r_i$, $i = \overline{1, m}$).

We introduce an analog of notions of least upper bound and greatest lower bound of a one-dimensional set for a bounded set $D \subset R^m$ [10, p. 11], [7, p. 32] without considering these bounds as elements of an ordered set of subsets of the space R^m. To this end, to each point $r \in R^m$, we assign the sets

$$\overline{K}(r) = \{p \in R^m : \ p \geq r\}, \quad \underline{K}(r) = \{p \in R^m : \ p \leq r\},$$

which are referred to as the *upper and lower direct m-dimensional angles*, respectively, *with vertex at the point r.*

Definition 2 ([9])**.** The *least upper* (respectively, *greatest lower*) *bound* of a set $D \subset R^m$ bounded above (respectively, below) is defined as the set $\sup D$ (respectively, $\inf D$) of vertices of all upper direct m-dimensional angles $\overline{K}(r)$ (respectively, lower direct m-dimensional angles $\underline{K}(r)$), each of which has the unique common point, the angle vertex, with the set $\overline{D}$,

$$\sup D \equiv \big\{r \in R^m : \ \overline{D} \cap \overline{K}(r) = \{r\}\big\} \quad \Big(\text{respectively}, \ \inf D \equiv \big\{r \in R^m : \ \overline{D} \cap \underline{K}(r) = \{r\}\big\}\Big).$$

Definition 3 ([9])**.** A set $D \subset R^m$ is said to be *upper closed* (respectively, *lower closed*) if it contains the least upper bound (respectively, the greatest lower bound) of itself.

Let the set $D \subset R^m$ be a connected upper and lower closed convex set. Note that the sets are its least upper bound $\sup D$ and greatest lower bound $\inf D$ have the properties of characteristic and lower characteristic sets, respectively.

Theorem. *Let sets $P \subset R^m$ and $\Lambda \subset R^m$ be defined, respectively, convex function $p_m = f_P(p_1, \dots, p_{m-1}) : R^{m-1} \to R$ and concave function $\lambda_m = f_\Lambda(\lambda_1, \dots, \lambda_{m-1}) : R^{m-1} \to R$ continuous monotonically decreasing in their convex closed bounded domain, and satisfy*

$$\sup \big\{p_i : \ (p_1, p_2, \dots, p_m) \in P\big\} \leq \inf \big\{\lambda_i : \ (\lambda_1, \lambda_2, \dots, \lambda_m) \in \Lambda\big\}, \quad i = \overline{1, m}.$$

Then there exists a completely integrable Pfaff equation

$$\frac{\partial x}{\partial t_i} = A_i(t)x, \ \ x \in R, \ \ t \in R^m_+, \ \ i = \overline{1, m}, \tag{1_2}$$

with bounded infinitely differentiable coefficient $A_i(t)$ with characteristic set $\Lambda(A) = \Lambda$ and lower characteristic set $P(A) = P$.

Sketch of the proof. Without loss of generality, one can assume (to within a shift) that the set $P \subset R^m$ lies in the m-dimensional cube $[d_1, d_2] \times \dots \times [d_1, d_2] \subset R^m_-$, and the set $\Lambda \subset R^m$ lies in the cube $[|d_2|, |d_1|] \times \dots \times [|d_2|, |d_1|] \subset R^m_+$, where $d_1 < d_2 \leq 0$.

I. Preliminary construction

Let us assume that the sets P and Λ, determines the functions f_P and f_Λ, admit the following parametric representation

$$P : \ \ p = H(\alpha) \ \text{ and } \ \Lambda : \ \ p = G(\alpha), \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{m-1}), \ \ \alpha_i \in [0, 1].$$

By the assumptions of the theorem, for each point of the sets P and $\Lambda \subset R^m$, there exists a tangent hyperplane, and if several tangent hyperplanes exist at some point of that set, then we take a hyperplane whose normal has coordinates of one sign. In addition, any of those tangent hyperplanes μ at the set $P \subset R^m$ lies not below that set, and any of those tangent hyperplanes ν at the set $\Lambda \subset R^m$ lies not above that set Λ. It means that for each $s \in P$, there exists $M_s \in \mu$ such that $s \leq M_s$, and for each $s \in \Lambda$, there exists $M_s \in \nu$ such that $s \geq M_s$. Let the tangent hyperplane μ of the set P at the point $H(\alpha)$ and the tangent hyperplane ν of the set Λ at the point $G(\alpha)$ be defined by the points $q^{(i)}(\alpha) \in R^m$ and $r^{(i)}(\alpha) \in R^m$, $i = \overline{1, m}$, respectively,

$$\begin{aligned}
\mu(\alpha, \zeta) &= q^{(1)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)(1 - \zeta_1) + q^{(2)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)\zeta_1 + \cdots \\
&\quad + q^{(m-1)}(\alpha) \cdot (1 - \zeta_{m-1})\zeta_{m-2} + q^{(m)}(\alpha) \cdot \zeta_{m-1}, \ \ \zeta = (\zeta_1, \zeta_2, \dots, \zeta_{m-1}), \ \ \zeta_i \in [0, 1], \\
\nu(\alpha, \zeta) &= r^{(1)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)(1 - \zeta_1) + r^{(2)}(\alpha) \cdot (1 - \zeta_{m-1}) \cdots (1 - \zeta_2)\zeta_1 + \cdots \\
&\quad + r^{(m-1)}(\alpha) \cdot (1 - \zeta_{m-1})\zeta_{m-2} + r^{(m)}(\alpha) \cdot \zeta_{m-1}, \ \ \zeta = (\zeta_1, \zeta_2, \dots, \zeta_{m-1}), \ \ \zeta_i \in [0, 1].
\end{aligned}$$

In this case, we set $q^{(1)}(\alpha) = H(\alpha)$, $r^{(1)}(\alpha) = G(\alpha)$ and require that the projections of those tangents $\mu(\alpha,\zeta)$ and $\nu(\alpha,\zeta)$ to the coordinate axes lies inside the corresponding projections of the sets P and Λ, respectively.

We construct the sequence $\{\tau_n^{(j)}(h)\}$, $h = (h_1, h_2, \dots, h_{m-1})$, where j for any fixed $n \in N$ ranges over the values $1, 2$, and h_i for fixed values of n, j, $h_1, \dots, h_{i-1}$ ranges over the values $1, \dots, 2^n$. We set the first element $\tau_1^{(1)}(1, \dots, 1)$ of that sequence to unity, and other elements obtained by multiplying by two the previous element of this sequence.

As a result, we obtain

$$\begin{aligned}\tau_n^{(j)}(h) &= 2^{2^{\sum\limits_{l=1}^{n}(2^{(l-1)})^{m-1}+(j-1)(2^n)^{m-1}+(h_1-1)(2^n)^{m-2}+\cdots+(h_{m-3}-1)(2^n)^2+(h_{m-2}-1)2^n+h_{m-1}-1}} \\ &\le \tau_{n+1}^{(1)}(1,\dots,1) = 2^{2^{\sum\limits_{l=1}^{n+1}(2^{(l-1)})^{m-1}}} \equiv 2^{\sigma_m(n)}.\end{aligned}$$

We set $\tau_t = t_1 + t_2 + \cdots + t_m$. We divide the subset $R_+^m = \{t = (t_1, t_2, \dots, t_m) :\ t_i \ge 0\}$ of the space R^m by the planes $\tau_t = 2^k$, $k \in N$, into the layers $\{t \in R_+^m :\ 2^k \le \tau_t < 2^{k+1}\}$, with the closed "lower" face and the open "upper" face. By $\Pi_0^{(1)}(1, \dots, 1)$ we denote the initial layer $\{t \in R_+^m :\ 0 \le \tau_t < \tau_1^{(1)}(1, \dots, 1)\}$. Next successively denote the layers by $\Pi_n^{(j)}(h)$, where j takes the values 1, 2 for a fixed $n \in N$, and h_i takes the values $1, \dots, 2^n$ for a fixed n, j, $h_1, \dots, h_{i-1}$. The lower part of the layer $\Pi_n^{(j)}(h)$ is defined as the layer

$$\widetilde{\Pi}_n^{(j)}(h) = \Big\{t \in \Pi_n^{(j)}(h) :\ \tau_n^{(j)}(h) \le \tau_t < \overline{\tau}_n^{(j)}(h)\Big\},$$

where

$$\overline{\tau}_n^{(j)}(h) \equiv \tau_n^{(j)}(h)\sqrt{2}\,,$$

and the top part is defined as the layer

$$\widetilde{\widetilde{\Pi}}_n^{(j)}(h) = \Big\{t \in \Pi_n^{(j)}(h) :\ \overline{\tau}_n^{(j)}(h) \le \tau_t < \overline{\tau}_n^{(j)}(h)\sqrt{2}\Big\}.$$

Following [4], [9], on the segment $\Delta_0^{(1)} = [0,1]$ we construct perfect set

$$P_0 = \bigcap_{n=1}^{+\infty} \bigcup_{k=1}^{2^n} \Delta_n^{(k)},$$

similar to the Cantor perfect set [8, p. 50] with a nonzero Lebesgue measure and modified step functions $\Theta(\alpha)$ [8, p. 200]. Wherein the length of the nst rank segments $\Delta_n^{(k)}$ will be assumed equal $\varepsilon_n = \exp(d_1 \cdot 2^{\sigma_m(n)})$, and the middle of these segments will be denoted $\alpha_n^{(k)}$. Next on the segment $\Delta_0^{(1)} = [0,1]$ we define continuous nondecreasing Cantor step function $\Theta(\alpha) :\ \Delta_0^{(1)} \to [0,1] = \{\Theta(\alpha) :\ \alpha \in P_0\}$ with intervals $\delta_n^{(k)} = \Delta_n^{(k)} \setminus (\Delta_{n+1}^{(2k-1)} \cup \Delta_{n+1}^{(2k)})$ of constant values.

Note that by the definition of P_0 for all the $n \in N$ there exists a number $k = k^{(n)}(\alpha) \in \{1, \dots, 2^n\}$, for which the inequality $|\alpha_n^{(k)} - \alpha| \le \varepsilon_n/2$, $k = k^{(n)}(\alpha)$, $n \in N$. Therefore we have $\Theta\big(\alpha_n^{(k_n(\alpha))}\big) \to \alpha$ if $n \to \infty$. We introduce the notation $\Theta(\alpha, h) \equiv (\Theta(\alpha_n^{(h_1)}), \dots, \Theta(\alpha_n^{(h_{m-1})}))$, $n \in N$.

II. Construction of the equation

For further constructions, we use the following functions infinitely differentiable on the interval $[\tau_1, \tau_2]$:

$$e_{01}(\tau,\tau_1,\tau_2)=\begin{cases}\exp\big\{-[\tau-\tau_1]^{-2}\exp\big(-[\tau-\tau_2]^{-2}\big)\big\} & \text{if } \tau_1<\tau<\tau_2,\\ i-1 & \text{if } \tau=\tau_i,\ \ i=1,2,\end{cases}$$

$$e_{00}(\tau,\tau_1,\tau_2)=\begin{cases}\exp\big(2^4(\tau_2-\tau_1)^{-4}-(\tau-\tau_1)^{-2}(\tau-\tau_2)^{-2}\big) & \text{if } \tau_1<\tau<\tau_2,\\ 0 & \text{if } \tau=\tau_i,\ \ i=1,2,\end{cases}$$

these are analogs of standard functions infinitely differentiable on the segment $[0,1]$. Note that the function $e_{00}(\tau,\tau_1,\tau_2)$ attains its maximum value unity in the middle of the segment $[\tau_1,\tau_2]$. On the sets

$$\Pi^{(1)}\equiv\bigcup_{n=1}^{+\infty}\bigcup_{h_1=1}^{2^n}\cdots\bigcup_{h_{m-1}=1}^{2^n}\Pi_n^{(1)}(h)$$

and

$$\Pi^{(2)}\equiv\bigcup_{n=1}^{+\infty}\bigcup_{h_1=1}^{2^n}\cdots\bigcup_{h_{m-1}=1}^{2^n}\Pi_n^{(2)}(h),$$

we introduce the vector functions

$$\mathcal{Q}^{(i)}(\tau_t)=\begin{cases}0 & \text{if } t\in\widetilde{\Pi}_n^{(j)}(h),\\ q^{(i)}(\Theta(\alpha,h))e_{00}\Big(\dfrac{\tau_t}{\overline{\overline{\tau}}_n^{(j)}}(h),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\widetilde{\Pi}}_n^{(j)}(h),\end{cases}\qquad i=\overline{1,m},$$

$$\mathcal{R}^{(i)}(\tau_t)=\begin{cases}0 & \text{if } t\in\widetilde{\Pi}_n^{(j)}(h),\\ r^{(i)}(\Theta(\alpha,h))e_{00}\Big(\dfrac{\tau_t}{\overline{\overline{\tau}}_n^{(j)}}(h),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\widetilde{\Pi}}_n^{(j)}(h),\end{cases}\qquad i=\overline{1,m}.$$

We introduce the functions

$$\mathcal{E}(t)=e^{(\mathcal{Q}^{(1)}(\tau_t),t)}+e^{(\mathcal{Q}^{(2)}(\tau_t),t)}+\cdots+e^{(\mathcal{Q}^{(m)}(\tau_t),t)}\ \ \text{if } t\in\Pi^{(1)},$$

$$E(t)=\big[e^{-(\mathcal{R}^{(1)}(\tau_t),t)}+e^{-(\mathcal{R}^{(2)}(\tau_t),t)}+\cdots+e^{(\mathcal{R}^{(m)}(\tau_t),t)}\big]^{-1}\ \ \text{if } t\in\Pi^{(2)}.$$

Obviously, the function $\mathcal{E}(t)$ takes a value equal to m if $t\in\widetilde{\Pi}_n^{(1)}(h)$, and the function $E(t)$ takes a value equal to m^{-1} if $t\in\widetilde{\Pi}_n^{(2)}(h)$. We construct the function $x(t)$, $t\in R_+^m$, by the following rule

$$x(t)=\begin{cases}m^{-1}+[m-m^{-1}]e_{01}\Big(\dfrac{\tau_t}{\tau_n^{(j)}}(h),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\Pi}_n^{(1)}(1,1,\dots,1),\\ \mathcal{E}(t) & \text{if } t\in\Pi^{(1)}\setminus\widetilde{\Pi}_n^{(1)}(1,1,\dots,1),\\ m+[m^{-1}-m]e_{01}\Big(\dfrac{\tau_t}{\tau_n^{(j)}}(1,1,\dots,1),1,\sqrt{2}\Big) & \text{if } t\in\widetilde{\Pi}_n^{(2)}(1,1,\dots,1),\\ E(t) & \text{if } t\in\Pi^{(2)}\setminus\widetilde{\Pi}_n^{(2)}(1,1,\dots,1).\end{cases}$$

This function is infinitely differentiable and is a solution of the Pfaff equation (1_2) with bounded infinitely differentiable on R_+^m coefficients

$$A_i(t)=x^{-1}(t)\,\frac{\partial x(t)}{\partial t_i}\,.$$

The infinite differentiability of $A_i(t)$ follows from the similar property of the functions, through which they are defined. Boundedness of coefficients $A_i(t)$ easy to show with the help of estimates given in [5] for functions $\frac{de_{01}(\tau,\tau_1,\tau_2)}{d\tau}$ and $\frac{de_{00}(\tau,\tau_1,\tau_2)}{d\tau}$, defined on any interval $[\tau_1,\tau_2]$ of length $\tau_2-\tau_1\le 1/2$.

III. Computation of the characteristic sets

Using conditions (2) and (3), the definition of the characteristic and the lower characteristic vectors, and the obvious estimates

$$\ln \mathcal{E}(t) > \max_{i\in\{1,2,\dots,m\}} \big\{(\mathcal{Q}^{(i)}(\tau_t),t)\big\}, \quad \ln E(t) < \min_{i\in\{1,2,\dots,m\}} \big\{(\mathcal{R}^{(i)}(\tau_t),t)\big\},$$

can be shown that the characteristic set of functions $x(t)$ is the set $\Lambda = \Lambda_E$, and the lower characteristic set of functions $x(t)$ is the set $P = P_{\mathcal{E}}$.

Comment

The result for equation (1_2) is easy to transfer on system (1).

References

[1] I. V. Gaĭshun, Completely integrable multidimensional differential equations. (Russian) *"Navuka i Tekhnika", Minsk*, 1983.

[2] I. V. Gaĭshun, Linear total differential equations. (Russian) *"Nauka i Tekhnika", Minsk*, 1989.

[3] È. I. Grudo, Characteristic vectors and sets of functions of two variables and their fundamental properties. (Russian) *Differencial'nye Uravnenija* **12** (1976), no. 12, 2115–2118.

[4] N. A. Izobov, On the existence of linear Pfaffian systems whose set of lower characteristic vectors has a positive plane measure. (Russian) *Differ. Uravn.* **33** (1997), no. 12, 1623–1630; translation in *Differential Equations* **33** (1997), no. 12, 1626–1632 (1998).

[5] N. A. Izobov, S. G. Krasovskiĭ, and A. S. Platonov, Existence of linear Pfaffian systems with lower characteristic set of positive measure in the space $\mathbb{R}^3$. (Russian) *Differ. Uravn.* **44** (2008), no. 10, 1311–1318; translation in *Differ. Equ.* **44** (2008), no. 10, 1367–1374.

[6] N. A. Izobov and A. S. Platonov, Construction of a linear Pfaff equation with arbitrarily given characteristics and lower characteristic sets. (Russian) *Differ. Uravn.* **34** (1998), no. 12, 1596–1603, 1725; translation in *Differential Equations* **34** (1998), no. 12, 1600–1607 (1999).

[7] A. N. Kolmogorov and S. V. Fomin, Elements of the theory of functions and functional analysis. (Russian) *Izdat. "Nauka", Moscow*, 1976.

[8] I. P. Natanson, Theory of functions of a real variable. (Russian) *Izdat. "Nauka", Moscow*, 1974.

[9] A. S. Platonov and S. G. Krasovskii, Existence of a linear Pfaff system with arbitrary bounded disconnected lower characteristic set of positive Lebesgue m-measure. *Differential Equations* **52** (2016), no. 10, 1300-1311.

[10] H. H. Schaefer, Topological vector spaces. *Graduate Texts in Mathematics*, Vol. 3. *Springer-Verlag, New York–Berlin*, 1971.