Non-Oscillation Criteria for Two-Dimensional System of Nonlinear Ordinary Differential Equations

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On the half-line $\mathbb{R}_+ = [0, +\infty[$, we consider the two-dimensional system of nonlinear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}} \operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha} \operatorname{sgn} u, \end{aligned} \tag{1}$$

where $\alpha > 0$ and $p, g : \mathbb{R}_+ \to \mathbb{R}$ are locally Lebesgue integrable functions such that

$$g(t) \geq 0 \quad \text{for a.e. } t \geq 0. \tag{2}$$

By a solution of system (1) on the interval $J \subseteq [0, +\infty[$ we understand a pair (u, v) of functions $u, v : J \to \mathbb{R}$, which are absolutely continuous on every compact interval contained in J and satisfy equalities (1) almost everywhere in J.

Definition 1. A solution (u, v) of system (1) is called *non-trivial* if $|u(t)| + |v(t)| \neq 0$ for $t \geq 0$. We say that a non-trivial solution (u, v) of system (1) is *non-oscillatory* if at least one of its component does not have a sequence of zeros tending to infinity.

Remark 2. It was proved by Mirzov in [11] that all non-extendable solutions of system (1) are defined on the whole interval $[0, +\infty[$. Therefore, when we are speaking about a solution of system (1), we assume that it is defined on $[0, +\infty[$. Moreover, in [11, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (1) if the function g is nonnegative. Especially, under assumption (2), if system (1) has a non-oscillatory solution, then any other its non-trivial solution is also non-oscillatory. Consequently, it is possible to introduce the following definition.

Definition 3. We say that system (1) is *non-oscillatory* if all its non-trivial solutions are non-oscillatory.

Oscillation and non-oscillation theory for ordinary differential equations and their systems is a widely studied topic of the qualitative theory of differential equation. Below presented results are closely related to those which are established in [1, 2, 4–10, 12, 13]. Some criteria stated in these papers are generalized below.

Indeed, one can see that system (1) is a generalization of the equation

$$u'' + \frac{1}{\alpha}\, p(t)|u|^{\alpha}|u'|^{1-\alpha} \operatorname{sgn} u = 0, \tag{3}$$

where $\alpha \in]0, 1]$ and $p : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. This equation is studied in the existing literature and some oscillation and non-oscillation criteria for equation (3) can be found, e.g., in [5, 8].

Moreover, many results (see, e.g., survey given in [2]) are known in the non-oscillation theory for the so-called "half-linear" equation

$$\big(r(t)|u'|^{q-1}\operatorname{sgn} u'\big)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0, \tag{4}$$

where $q > 1$, $p, r : [0, +\infty[\to \mathbb{R}$ are continuous and r is positive. It is clear that (4) is a particular case of system (1). Indeed, if the function u, with the properties $u \in C^1$ and $r|u'|^{q-1}\operatorname{sgn} u' \in C^1$, is a solution of equation (4), then the vector function $(u, r|u'|^{q-1}\operatorname{sgn} u')$ is a solution of system (1) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q - 1$.

However, there are some restrictions on functions p and g in the above-mentioned papers. It is usually assumed that $p(t) \geq 0$ or $\int\limits_0^t p(s)\,ds > 0$ for large t. Moreover, the coefficient $g(t) := r^{\frac{1}{1-q}}(t)$ of the half-linear equation (4) cannot have zero points in any neighbourhood of infinity. Below we formulate criteria without these additional assumptions.

We consider two different cases, when the coefficient g is non-integrable and integrable on the half-line.

a) The case $\int\limits_0^{+\infty} g(s)\,ds = +\infty$

At first, we assume that

$$\int\limits_0^{+\infty} g(s)\,ds = +\infty, \tag{5}$$

and we put

$$f(t) := \int\limits_0^t g(t)\,ds \quad \text{for } \ t \geq 0.$$

In view of assumptions (2) and (5), there exists $t_g \geq 0$ such that $f(t) > 0$ for $t > t_g$ and $f(t_g) = 0$. We can assume without loss of generality that $t_g = 0$, since we are interested in the behaviour of solutions in the neighbourhood of $+\infty$, i.e., we have

$$f(t) > 0 \quad \text{for } \ t > 0$$

and, moreover,

$$\lim_{t\to+\infty} f(t) = +\infty.$$

We put

$$c_\alpha(t) := \frac{\alpha}{f^\alpha(t)} \int\limits_0^t \frac{g(s)}{f^{1-\alpha}(s)} \bigg(\int\limits_0^s p(\xi)\,d\xi \bigg)\,ds \quad \text{for } \ t > 0.$$

It is known (see [3, Corollary 2.5 (with $\nu = 1 - \alpha$)]) that if a finite limit of the function $c_\alpha(t)$ does not exist and $\liminf\limits_{t\to+\infty} c_\alpha(t) > -\infty$, then system (1) is oscillatory. Consequetly, in what follows it is natural to assume that

$$\lim_{t\to+\infty} c_\alpha(t) =: c^*_\alpha \in \mathbb{R}. \tag{6}$$

We put

$$Q(t;\alpha) := f^\alpha(t)\bigg(c^*_\alpha - \int\limits_0^t p(s)\,ds\bigg) \quad \text{for } \ t > 0,$$

where the number c_α^* is given by (6). Moreover, we denote lower and upper limits of the function $Q(\,\cdot\,;\alpha)$ as follows

$$Q_*(\alpha) := \liminf_{t\to+\infty} Q(t;\alpha), \quad Q^*(\alpha) := \limsup_{t\to+\infty} Q(t;\alpha).$$

Theorem 4. *Let* (6) *hold. Let, moreover, the inequalities*

$$-\frac{2\alpha+1}{\alpha+1}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha} < Q_*(\alpha) \quad \textit{and} \quad Q^*(\alpha) < \frac{1}{\alpha+1}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}$$

be satisfied. Then system (1) *is nonoscillatory.*

We denote by $B(\xi)$ the greatest root of the equation

$$|x|^{\frac{\alpha}{\alpha+1}} + x + \xi = 0,$$

where $\xi \leq 0$. Now we can formulate the next theorem which complements the previous one in a certain sense.

Theorem 5. *Let* (6) *hold. Let, moreover, the inequalities*

$$-\infty < Q_*(\alpha) \leq -\frac{2\alpha+1}{\alpha+1}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}$$

and

$$Q^*(\alpha) < [B(Q_*(\alpha))]^{\frac{\alpha}{\alpha+1}} - B(Q_*(\alpha))$$

be satisfied. Then system (1) *is nonoscillatory.*

b) The case $\int\limits_0^{+\infty} g(s)\,ds < +\infty$

Now we assume that the coefficient g is integrable on $[0,+\infty[$, i.e.,

$$\int\limits_0^{+\infty} g(s)\,ds < +\infty.$$

Let

$$\widetilde{f}(t) := \int\limits_t^{+\infty} g(t)\,ds \quad \text{for } t \geq 0.$$

In view of assumptions (2) and (5), we have

$$\lim_{t\to+\infty} \widetilde{f}(t) = 0$$

and

$$\widetilde{f}(t) > 0 \quad \text{for } t \geq 0.$$

We put

$$\widetilde{c}_\alpha(t) := \widetilde{f}(t) \int\limits_0^t \frac{g(s)}{\widetilde{f}^2(s)} \bigg(\int\limits_0^s \widetilde{f}^{\alpha+1}(\xi) p(\xi)\,d\xi \bigg)\,ds \quad \text{for } t \geq 0.$$

According to [3, Corollary 2.11 (with $\nu = 1 - \alpha$)], the system (1) is oscillatory if function $\widetilde{c}_\alpha(t)$ does not have a finite limit and $\liminf\limits_{t\to+\infty} \widetilde{c}_\alpha(t) > -\infty$. Consequently, we assume that there exists a finite limit of the function $\widetilde{c}_\alpha$, i.e.,

$$\lim_{t\to+\infty} \widetilde{c}_\alpha(t) =: \widetilde{c}_\alpha^* \in \mathbb{R}.$$

We denote

$$\widetilde{Q}(t;\alpha) := \frac{1}{\widetilde{f}(t)} \Big(\widetilde{c}_\alpha^* - \int_0^t \widetilde{f}^{\alpha+1}(s) p(s)\, ds \Big) \quad \text{for } t > 0.$$

Moreover, we denote lower and upper limits of the functions $\widetilde{Q}(\,\cdot\,;\alpha)$ as follows

$$\widetilde{Q}_*(\alpha) := \liminf_{t\to+\infty} \widetilde{Q}(t;\alpha), \quad \widetilde{Q}^*(\alpha) := \limsup_{t\to+\infty} \widetilde{Q}(t;\alpha).$$

Now we formulate next nonoscilation criteria by using lower and upper limits of the function $\widetilde{Q}(t;\alpha)$. We denote by $\widetilde{A}(\nu)$ and $\widetilde{B}(\nu)$ the smallest and the greatest root of the equation

$$\alpha|x|^{\frac{\alpha+1}{\alpha}} + (\alpha+1)x + \nu = 0.$$

Theorem 6. *Let the inequalities*

$$\widetilde{A}(\nu) + \nu < \widetilde{Q}_*(\alpha) \quad \text{and} \quad \widetilde{Q}^*(\alpha) < \Big(\frac{\alpha}{\alpha+1} \Big)^{\alpha+1}$$

be fulfilled with $\nu = \frac{2\alpha+1}{\alpha+1} \big(\frac{\alpha}{1+\alpha} \big)^{1+\alpha}$. *Then system* (1) *is nonoscillatory.*

The following theorem complements previous one in a certain sense. Before we formulate it, we denote by $\widehat{B}(\eta)$ the greatest root of the equation

$$\alpha|x|^{\frac{\alpha+1}{\alpha}} - \alpha x + \eta = 0,$$

where $\eta < \big(\frac{\alpha}{\alpha+1} \big)^{\alpha+1}$.

Theorem 7. *Let the inequalities*

$$-\infty < \widetilde{Q}_*(\alpha) \le \widetilde{A}(\nu) + \nu$$

with $\nu = \frac{2\alpha+1}{\alpha+1} \big(\frac{\alpha}{1+\alpha} \big)^{1+\alpha}$, *and*

$$\widetilde{Q}^*(\alpha) < \widetilde{Q}_*(\alpha) + \widehat{B}(\widetilde{Q}_*(\alpha)) + \widetilde{B}\Big(\widetilde{Q}_*(\alpha) + \widehat{B}(\widetilde{Q}_*(\alpha)) \Big)$$

be satisfied. Then system (1) *is nonoscillatory.*

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