Approximation of the Optimal Control Problem on an Interval with a Family of Optimization Problems on time Scales

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This work is devoted to the study of the limiting behavior of the optimal control problem for dynamic equations defined on a family of time scales $\mathbb{T}_\lambda$, in the regime when the graininess function μ_λ converges to zero as $\lambda \to 0$. At the same time the segment of the time scale $[t_0, t_1]_{\mathbb{T}_\lambda} = [t_0, t_1] \cap \mathbb{T}_\lambda$ approaches $[t_0, t_1]$ e.g. in the Hausdorff metric. The natural question that arises is how the optimal control problem on the time scale is related to the corresponding control problem on the interval $[t_0, t_1]$.

The time scales theory was introduced by S. Hilger [6] (1988) as a unified theory for both discrete and continuous analysis. For reader's convenience, we present several notions from this theory which are used in this paper.

Time scale $\mathbb{T}$ is a non-empty closed subset of $\mathbb{R}$, $A_{\mathbb{T}} := A \cap \mathbb{T}$ for $A \subset \mathbb{R}$, $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$ is the forward jump operator, $\rho : \mathbb{T} \to \mathbb{T}$, $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$ is the backward jump operator (here $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$), $\mu : \mathbb{T} \to [0, \infty)$, $\mu(t) := \sigma(t) - t$ is called the graininess function. A point $t \in \mathbb{T}$ is called left-dense (LD) (left-scattered (LS), right-dense (RD) or right-scattered (RR)) if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$), $\mathbb{T}^k := \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has a left-scattered maximum M, $\mathbb{T}^k := \mathbb{T}$ otherwise.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is called Δ-differentiable at $t \in \mathbb{T}^k$ if the limit

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$.

Let $\Lambda \subset \mathbb{R}$, such that 0 is a limit point of Λ, be the set of indices. Consider the family of time scales $\mathbb{T}_\lambda, \lambda \in \Lambda$ such that $\sup \mathbb{T}_\lambda = \infty$. For any $t_0, t_1 \in \mathbb{T}_\lambda$ denote $[t_0, t_1]_{\mathbb{T}_\lambda} = [t_0, t_1] \cap \mathbb{T}_\lambda$ and $\mu_\lambda = \sup_{t \in [t_0, t_1]_{\mathbb{T}_\lambda}} \mu(t)$. Assume

$$\mu_\lambda(t) \to 0 \ \text{ as } \ \lambda \to 0. \tag{1}$$

For every $\mathbb{T}_\lambda$ consider the optimal control problem on the time scale $[t_0, t_1]_{\mathbb{T}_\lambda}$:

$$\begin{cases} x^\Delta = f(t, x, u), \\ x(t_0) = x, \\ J_\lambda(u) = \displaystyle\int\limits_{[t_0, t_1)_{\mathbb{T}_\lambda}} L(t, x(t), u(t))\, \Delta t + \Psi(x(t_1)) \longrightarrow \inf, \ \ u \in \mathcal{U}(t_0). \end{cases} \tag{2}$$

Along with (2), consider the corresponding continuous optimal control problem on the interval $[t_0, t_1]$:

$$\begin{cases} \dfrac{dx(t)}{dt} = f\big(t, x(t), u(t)\big), \\ x(t_0) = x, \\ J(u) = \displaystyle\int_{t_0}^{t_1} L\big(t, x(t), u(t)\big)\, dt + \Psi(x(t_1)) \longrightarrow \inf, \quad u \in \mathcal{U}(t_0), \end{cases} \tag{3}$$

where $x \in \mathbb{R}^d$, $u \in U \subset \mathbb{R}^m$, U – compact set, $\mathcal{U}(t_0) := L^\infty([t_0, t_1]_{\mathbb{T}}, U)$, i.e. the set of bounded, Δ – measurable functions [2, Chapter 5.7] defined on $[t_0, t_1]_{\mathbb{T}}$ and taking values in U for each $t \in [t_0, t_1]_{\mathbb{T}}$, is called the set of admissible controls.

Assume that f, L and Ψ satisfy

(i) $f : [t_0, t_1]_{\mathbb{T}} \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $L : [t_0, t_1]_{\mathbb{T}} \times \mathbb{R}^d \times U \to \mathbb{R}^1$ and $\Psi : \mathbb{R}^d \to \mathbb{R}^1$;

(ii) f is continuous and globally Lipschitz in x with the Lipschitz constant K;

(iii) L and Ψ are continuous and globally Lipschitz in x with the Lipschitz constant K.

The Bellman function in this case is

$$V(t_0, x) := \inf_{u(\,\cdot\,) \in \mathcal{U}(t_0)} J(t_0, x, u). \tag{4}$$

Denote by $V_\lambda(t_0, x)$ and $V(t_0, x)$ the corresponding Bellman functions for these problems, given by (4). Our main result is the following theorem.

Theorem 1. *Let $\mathbb{T}_\lambda$ be such that (1) holds. In addition, assume that*

1) *The functions f, f_x and L are continuous on $[t_0, t_1] \times \mathbb{R}^d \times U$;*

2) *f and L are globally Lipschitz in x, with Lipschitz constant $K > 0$.*

Then

$$V_\lambda(t_0, \,\cdot\,) \to V(t_0, \,\cdot\,) \ \text{ in } \ C_{loc}(\mathbb{R}^d), \quad \lambda \to 0.$$

The proof of the main result will heavily rely on two lemmas.

Without loss of generality, we assume that $t_0 = 0$ and $t_1 = 1$. Consider an arbitrary time scale $\mathbb{T}_\lambda$ and an arbitrary admissible control $u_\lambda(t)$ on it. Let $x_\lambda(t)$ be a corresponding admissible trajectory. Denote by $\widetilde{u}_\lambda(t)$ the extension of $u_\lambda(t)$ to the entire interval $[0, 1]$:

$$\widetilde{u}_\lambda(t) := \begin{cases} u_\lambda(t), & t \in [0, 1]_{\mathbb{T}_\lambda}, \\ u_\lambda(r), & t \in [r, \sigma(r)), \quad r \in \mathrm{RS}\,. \end{cases} \tag{5}$$

This control is admissible for the problem (3).

Lemma 1. *Let $x(t)$ be a solution of*

$$\begin{cases} \dfrac{dx}{dt} = f\big(t, x, \widetilde{u}_\lambda(t)\big), \\ x(0) = x_0. \end{cases}$$

Then

$$\bigg| \int_{[0,1)_{\mathbb{T}_\lambda}} L\big(t, x_\lambda(t), u_\lambda(t)\big)\, \Delta t - \int_0^1 L\big(t, x(t), \widetilde{u}_\lambda(t)\big)\, dt \bigg| \longrightarrow 0, \quad \lambda \to 0.$$

Let $u^\lambda_{ts}(\,\cdot\,)$ be an arbitrary admissible control for the problem (2) and $x^\lambda_{ts}(\,\cdot\,)$ be the corresponding trajectory. Similarly, let $x(\,\cdot\,)$ be an admissible trajectory of the problem (3) which corresponds to the admissible control $u(\,\cdot\,)$.

Lemma 2. *For any admissible control $u(\,\cdot\,)$ for the problem* (3) *and for every time scale $\mathbb{T}_\lambda$, there is an admissible control $u^\lambda_{ts}(\,\cdot\,)$ for the problem* (2) *such that*

$$\big|J(u) - J_\lambda(u^\lambda_{ts})\big| \longrightarrow 0, \;\; \lambda \to 0.$$

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