An Estimate for Solutions to a Uniformly Charged Functional Differential Equation with Full Memory

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1 Introduction

Here we consider a class of functional differential systems that arises under attempts to reduce functional differential systems with continuous and discrete times [3] to equations with only continuous time having in mind to apply some results from the theory of functional differential equations [2]. First we recall the description of a class of continuous-discrete functional differential equations with linear Volterra operators and appropriate spaces where those are considered. Then a continuous-discrete system is reduced to a continuous system that turns out to be a charged functional differential system with a full memory. For this system, an estimate of solutions, which can be useful for analysis of their properties, is obtained.

2 Preliminaries

To describe the continuous subsystem, let us introduce the linear operator $\mathcal{L}$:

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int\limits_0^t K(t,s)\dot{x}(s)\,ds + A(t)x(0), \ \ t \in [0,T]. \tag{1}$$

Here the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $\{(t,s):\ 0 \leqslant s \leqslant t < \infty\}$ and such that

$$|k_{ij}(t,s)| \leqslant \kappa(t), \ \ i,j = 1,\dots,n,$$

where function κ is summable on $[0,T]$ for any finite $T > 0$, the elements $(n \times n)$-matrix A are summable on $[0,T]$ for any finite $T > 0$. By $AC^n[0,T]$ we denote the space of absolutely continuous functions $x:[0,T] \to R^n$, $L^n[0,T]$ denotes the space of functions Lebesgue summable on $z:[0,T] \to R^n$,

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}, \quad \|z\|_{L^n} = \int\limits_0^T |z(t)|\,dt,$$

where $|\alpha| = \max\limits_{i=1,\dots,n} |\alpha_i|$ for $\alpha = col(\alpha_1,\dots,\alpha_n) \in R^n$ (we reserve $\|\cdot\|$ for the corresponding norm in R^n). The operator $\mathcal{L}: AC^n[0,T] \to L^n[0,T]$ is bounded. The theory of equation $\mathcal{L}x = f$ is thorouhgly treated in [2, 6]. The equation $\mathcal{L}x = f$ covers differential equations with concentrated and/or distributed delay and integrodifferential Volterra equations. The Cauchy problem

$$\mathcal{L}x = f, \quad x(0) = \alpha$$

is uniquely solvable for any $f \in L^n[0,T]$ and $\alpha \in R^n$ and its solution has the representation

$$x(t) = X(t)\alpha + \int\limits_0^t C_1(t,s)f(s)\,ds,$$

where $X(\,\cdot\,)$ is the fundamental matrix, $C_1(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix [5].

For description of the discrete subsystem, we introduce the operator Λ:

$$(\Lambda y)(t_i) = y(t_i) - \sum_{j<i} B_{ij} y(t_j), \;\; i = 1,2,\dots,\mu, \;\; 0 = t_0 < t_1 < \dots < t_\mu = T.$$

Here B_{ij} are constant $(\nu\times\nu)$-matrices. Denote $J = \{t_0, t_1, \dots, t_\mu\}$, $FD^\nu(\mu)$ is the space of functions $y : J \to R^\nu$ normed by $\|y\|_{FD^\nu(\mu)} = \sum\limits_{i=0}^{\mu} |y(t_i)|$. Recall some facts on equation $\Lambda y = g$ (see, for instance, [1]). The Cauchy problem

$$\Lambda y = g, \quad y(0) = \beta$$

is uniquely solvable for any $g \in FD^\nu(\mu)$ $\beta \in R^\nu$ and its solution has the form

$$y(t_i) = Y(t_i)\beta + \sum_{j\leqslant i} C_2(i,j)g(t_j), \;\; i = 1,2,\dots,\mu, \tag{2}$$

where $Y(\,\cdot\,)$ is the fundamental matrix, $C_2(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

Consider the system

$$(\mathcal{L}x)(t) = \sum_{j:\, t_j<t} U_j(t)y(t_j) + f(t), \;\; t \in [0,T], \tag{3}$$

$$(\Lambda y)(t_i) = \sum_{j: t_j<t_i} A_{ij}x(t_j) + g(t_i), \;\; i = 1,2,\dots,\mu, \tag{4}$$

that consists of subsystem (3) with continuous time and subsystem (4) with discrete time. Here A_{ij} are constant matrices of dimension $\nu \times n$, U_j are $(n \times \nu)$-matrices with summable elements. The subsystems are connected between each other with respect their states.

3 A charged functional differential system

To reduce system (3), (4) to an equation with respect to $x(\,\cdot\,)$, we solve (4) with respect to $y(\,\cdot\,)$ by means of (2):

$$y(t_i) = Y(t_i)y(t_0) + \sum_{j\leqslant i} C_2(i,j)\Big(\sum_{j:\, t_\ell<t_j} A_{j\ell}x(t_\ell)\Big) + \sum_{j\leqslant i} C_2(i,j)g(t_j), \;\; i = 1,2,\dots,\mu,$$

and then substitute the right-hand side of the latter into (3). After immediate calculations subsystem (3) can be rewritten in the form of a charged (by the terms $V_j(t)x(t_j)$) functional differential equation

$$(\mathcal{L}x)(t) = \sum_{j: t_j<t} V_j(t)x(t_j) + r(t), \;\; t \in [0,T].$$

In the sequel, we consider this equation in the case $t_j = j$ and assume that T is as great as we wish:

$$(\mathcal{L}x)(t) = \sum_{j<t} V_j(t)x(j) + r(t), \;\; t \in [0,\infty). \tag{5}$$

Our aim is to obtain an estimate of solutions to (5). We derive this estimate on the base of the following Lemma that is a kind of the Gronwall-Bellman inequality.

Lemma. *Let $p(j)$, $q(j)$, $v(j)$, $z(j)$, $j = 0, 1, 2, \dots$ be nonnegative sequences such that*

$$z(j) \leq v(j) + p(j) \sum_{k=0}^{j-1} q(k) z(k), \quad k = 1, 2, \dots, \quad z(0) \leq v(0). \tag{6}$$

Then the estimate

$$z(j) \leq v(j) + p(j) \sum_{\ell=0}^{j-1} M_{j\ell} q(\ell) v(l), \quad j = 1, 2, \dots, \tag{7}$$

where

$$M_{j\ell} = \exp\Big(\sum_{i=\ell}^{j-1} p(i) q(i) \Big),$$

holds.

Remark. Let us note that, as to compare with the traditional version of (6), where $v(j) = cp(j)$, $c > 0$ and the estimate has the form

$$z(j) \leq cp(j) \prod_{\ell=0}^{j-1} \big(1 + p(\ell) q(\ell)\big) \tag{8}$$

(see, for instance, Corollary of Lemma 1.1 [4]), the estimate (7) can be much more sharp. Really, put $v(j) = 1 + 1/(1+j)$; $p(j) = 1/(1+j)$; $q(j) = 1/(1+j)^2$. By means of (7) we obtain $z(100) \leq 1.1$, whereas (8) gives $z(100) \leq 6.5$.

Denote

$$d_j = X(j) x_0 + \int_0^j C_1(j,s) r(s)\, ds, \quad D_{jk} = \int_k^j C_1(j,s) V_k(s)\, ds.$$

Theorem. *Let the following inequalities take place:*

$$|d_j| \leq v(j), \quad \|D_{jk}\| \leq p(j) q(k), \quad j, k = 1, 2, \dots, \quad k \leq j,$$

where $v(j)$, $p(j)$, $q(j)$, $j = 1, 2, \dots$ are nonnegative sequences. Then the estimate (7) *holds for $z(j) = |x(j)|$.*

Proof. First we use the representation of solutions to (1) as applied to (5):

$$x(t) = X(t) x_0 + \int_0^t C_1(t,s) r(s)\, ds + \int_0^t C_1(t,s) \sum_{k<s} V_k(s) x(k)\, ds, \quad t \in [0, T].$$

Thus, for sections $x(j)$, we have the system

$$x(j) = X(j) x_0 + \int_0^j C_1(j,s) r(s)\, ds + \int_0^j C_1(j,s) \sum_{k<s} V_k(s) x(k)\, ds. \tag{9}$$

Next note that the expression

$$\int_0^j C_1(j,s) \sum_{k<s} V_k(s) x(k)\, ds$$

can be written in the form
$$\sum_{k<j} D_{jk}x(k).$$
This follows from the immediate calculations. Denote
$$w(j)=X(j)x_0+\int\limits_0^j C_1(j,s)r(s)\,ds$$
and rewrite (9) in the form
$$x(t_j)=w(t_j)+\sum_{k<j} D_{jk}x(k). \tag{10}$$
To complete the proof, it remains to apply Lemma to the inequality
$$|x(j)|\le|w(j)|+\sum_{k<j}\|D_{jk}\|\,|x(k)|,$$
which follows from (10). □

This Theorem makes it possible to take into account asymptotic properties of the Cauchy matrix, the coefficients $V_j(t)$ as weights of the charges $x(j)$, and the free term $r(t)$ in (5) to answer questions about asymptotic behaviour of solutions. Here we restrict ourselves by the following example.

Example. Consider the linear charged differential equation
$$\dot{x}(t)+2tx(t)=\sum_{j<t} v_j(t)x(j)+r(t),\ \ t\in[0,\infty),$$
where $|v_j(t)|\le c\,\frac{1}{(1+j)^2}$. For this equation, the solution $x(t)$ with the initial condition $x(0)=x_0$ is bounded on $[0,\infty)$ for any $r(t)$ such that the inequality $|r(t)|\le d(1+t)$ holds with a $d>0$ almost everywhere on $[0,\infty)$, and the estimate
$$|x(j)|\le\Big(e^{-j^2}+\frac{11}{10}\,\frac{ce^{\frac{11}{5}c}}{\frac{e}{4}+j}\Big)|x_0|+\frac{3}{2}\,\Big(1+\frac{2ce^{\frac{11}{5}c}}{\frac{e}{4}+j}\Big)d,\ \ j=1,2,\dots$$
holds.

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