Some Properties of Minimal Malkin Estimates

E. K. Makarov

Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: jcm@im.bas-net.by

Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with a bounded piecewise continuous coefficient matrix A and the Cauchy matrix X_A. Suppose that $\|A(t)\| \leq a < +\infty$ for all $t \geq 0$. In [8], see also [9, p. 379] and [1, p. 236], I. G. Malkin has used estimations of the form

$$\|X_A(t,s)\| \leq D\exp(\alpha(t-s)+\beta s), \ \ t \geq s \geq 0, \ \ D > 0, \ \ \alpha, \beta \in \mathbb{R}, \tag{2}$$

in order to investigate asymptotic stability of the trivial solution to a system

$$\dot{y} = A(t)y + f(t,y), \ \ y \in \mathbb{R}^n, \ \ t \geq 0,$$

with a nonlinear perturbation $f(t,y)$ of a higher order. An ordered pair $(\alpha,\beta) \in \mathbb{R}^2$ is called a Malkin estimation for system (1) if there exists a number $D = D(\alpha,\beta) > 0$ such that (2) holds. We denote the set of all Malkin estimations for system (1) by $E(A)$.

A pair $(\alpha,\beta) \in \mathbb{R}^2$ is said to be a minimal Malkin estimation [7] if $(\alpha+\xi,\beta+\eta) \in E(A)$ for all $\xi > 0$, $\eta > 0$, and $(\alpha+\xi,\beta+\eta) \notin E(A)$ for all $\xi \leq 0$, $\eta \leq 0$, $\xi^2+\eta^2 \neq 0$. Note that a minimal Malkin estimation is not necessarily an element of $E(A)$ by definition; an example is given below. On the other hand, if $(\alpha,\beta) \in E(A)$ and numbers ξ and η are nonnegative, then the pair $(\alpha+\xi,\beta+\eta)$ satisfies inequality (2) with the same $D = D(\alpha,\beta)$ since $t \geq s \geq 0$, i.e. the inclusion $(\alpha+\xi,\beta+\eta) \in E(A)$ is now valid.

We denote the set of all minimal Malkin estimations for system (1) by $M(A)$.

It can be easily seen that the set of minimal Malkin estimations for system (1) coincides with the set of Grudo characteristic vectors [2] for the function $\|X_A(t,s)\|$ with respect to the cone $C = \{(t,s) \in \mathbb{R}^2 : \ t \geq s \geq 0\}$. Using this fact and the results of [2] we can give [7] another description for the set $M(A)$. Let $K = \{(\alpha,\beta) \in \mathbb{R}^2 : \ \alpha > 0, \ \beta > 0\}$ be the positive cone of $\mathbb{R}^2$ and $\preccurlyeq$ be the partial order in $\mathbb{R}^2$ corresponding to K. Then $M(A)$ coincides with the set of all minimal with respect to $\preccurlyeq$ elements of $\operatorname{cl} E(A)$, where cl is the operator of closure.

The invariant uniform exponent $\iota[x]$ of a nonzero solution x to system (1) is the number $\sup N(x)$, where the set $N(x)$ consists of all numbers

$$\varlimsup_{k\to+\infty} \frac{1}{(t_k-s_k)} \ln \frac{\|x(t_k)\|}{\|x(s_k)\|}$$

such that the sequence of pairs $\tau_k = (t_k,s_k) \in \mathbb{R}^2$, $t_k \geq s_k \geq 0$, $k \in \mathbb{N}$, satisfy the condition $\inf\limits_k s_k^{-1}t_k > 1$ and $t_k - s_k \to +\infty$ as $k \to +\infty$.

The invariant general exponent $\mathrm{I}_0(A)$ for system (1) is the number

$$\mathrm{I}_0(A) = \sup_{\theta>0} \varlimsup_{s\to+\infty} \frac{1}{(\theta-1)s} \ln \|X_A(\theta s,s)\|. \tag{3}$$

These two exponents are invariant with respect to generalized Lyapunov transformations [3], whereas the analogous Bohl uniform and general exponents are not invariant.

There exists an alternative characterization for $\mathrm{I}_0(A)$ given in [7]. Namely, $\mathrm{I}_0(A)$ is the first component of a unique pair $(\alpha,0)\in M(A)$. It should be stressed that the pair $(\mathrm{I}_0(A),0)$ is always in $M(A)$, but the inclusion $(\mathrm{I}_0(A),0)\in E(A)$ is not valid in general. Indeed, according to [1, p. 109], [4, p. 68], and [5, p. 63] for any $\varepsilon>0$ we have

$$\|X_A(t,s)\|\le D_\varepsilon \exp\big((\Omega_0(A)+\varepsilon)(t-s)\big) \tag{4}$$

with some $D_\varepsilon>0$, where

$$\Omega_0(A)=\lim_{T\to+\infty}\,\overline{\lim_{k\to\infty}}\,T^{-1}\ln\big\|X_A(kT,kT-T)\big\| \tag{5}$$

is the general exponent of system (1). A similar estimation

$$\|X_A(t,s)\|\le D_\varepsilon \exp(\alpha(t-s)) \tag{6}$$

with $\alpha<\Omega_0(A)$ is not possible at all. Thus, $(\Omega_0(A)+\varepsilon,0)\in E(A)$ for each $\varepsilon>0$ and there are no pairs $(\alpha,0)\in E(A)$ with $\alpha<\Omega_0(A)$. On the other hand, from (3) and (5) we can assert that the inequality $\Omega_0(A)\ge \mathrm{I}_0(A)$ is always valid and that $\Omega_0(A)>\mathrm{I}_0(A)$ in general. Thereby $(\mathrm{I}_0(A),0)\notin E(A)$ in general too.

It was proved in [7] that the invariant general exponent $\mathrm{I}_0(A)$ is the attainable upper bound for invariant uniform exponents under exponentially small perturbations. Our aim is to obtain some similar interpretation for all elements of $M(A)$. To this end, we first obtain some alternative formulas for $\mathrm{I}_0(A)$ and $\iota[x]$.

Proposition 1. *For any system* (1) *the equalities*

$$\mathrm{I}_0(A)=\lim_{\theta\to1+0}\,\overline{\lim_{s\to+\infty}}\,\frac{1}{(\theta-1)s}\,\ln\|X_A(\theta s,s)\|=\lim_{\theta\to1+0}\,\overline{\lim_{k\to\infty}}\,\frac{1}{(\theta-1)\theta^k}\,\ln\big\|X_A(\theta^{k+1},\theta^k)\big\|$$

hold.

Proof. Let

$$R(\theta,s)=\frac{1}{(\theta-1)s}\,\ln\|X_A(\theta s,s)\|,\quad R(\theta)=\overline{\lim_{k\to\infty}}\,R(\theta,\theta^k),\quad \underline{\mathrm{I}}=\underline{\lim}_{\theta\to1+0}R(\theta).$$

Take any $\varepsilon>0$, $\theta>1$ and put $\vartheta=1+\varepsilon a^{-1}(\theta-1)/(\theta+1)$. By definition of lower limit, for any $\varepsilon>0$ and $\vartheta>1$ there exists a number $\theta_\varepsilon\in]1,\vartheta]$ such that the inequality $R(\theta_\varepsilon)<\underline{\mathrm{I}}+\varepsilon$ holds. Then by definition of upper limit, for the same $\varepsilon>0$ there exists a number $N_\varepsilon\in\mathbb{N}$ such that the inequality

$$R(\theta_\varepsilon,\theta_\varepsilon^j)<\overline{\lim_{j\to\infty}}\,R(\theta_\varepsilon,\theta_\varepsilon^j)+\varepsilon<\underline{\mathrm{I}}+2\varepsilon$$

is valid for each $j>N_\varepsilon$.

Take any $s>\theta_\varepsilon^{N_\varepsilon}$ and find numbers $p,q\in\mathbb{N}$ such that $s\in[\theta_\varepsilon^p,\theta_\varepsilon^{p-1}[$ and $\theta s\in[\theta_\varepsilon^{q+2},\theta_\varepsilon^{q+1}[$. Then we have

$$\begin{gathered}\theta_\varepsilon^p-s\le\theta_\varepsilon^p-\theta_\varepsilon^{p-1}=\theta_\varepsilon^{p-1}(\theta_\varepsilon-1)\le(\theta_\varepsilon-1)s,\\ \theta s-\theta_\varepsilon^{q+1}\le\theta_\varepsilon^{q+2}-\theta_\varepsilon^{q+1}=\theta_\varepsilon^{q+1}(\theta_\varepsilon-1)\le(\theta_\varepsilon-1)\theta s,\end{gathered}$$

and

$$
\begin{aligned}
(\theta-1)sR(\theta,s) &\le \ln \|X(\theta s,\theta_\varepsilon^{q+1})\| + \ln\|X(\theta_\varepsilon^p,s)\| + \sum_{j=p}^{q} \ln \|X(\theta_\varepsilon^{j+1},\theta_\varepsilon^j)\| \\
&\le a(\theta s-\theta_\varepsilon^{q+1}+\theta_\varepsilon^p-s) + \sum_{j=p}^{q}(\theta_\varepsilon^{j+1}-\theta_\varepsilon^j)R(\theta_\varepsilon,\theta_\varepsilon^j) \\
&\le as(\theta+1)(\theta_\varepsilon-1) + (\theta_\varepsilon^{q+1}-\theta_\varepsilon^p)\max_{q\le j\le p} R(\theta_\varepsilon,\theta_\varepsilon^j) \le as(\theta+1)(\vartheta-1)+(\theta-1)s\max_{q\le j\le p} R(\theta_\varepsilon,\theta_\varepsilon^j).
\end{aligned}
$$

By the above assumptions we have

$$
R(\theta,s) \le a(\theta+1)(\vartheta-1)/(\theta-1) + \max_{q\le j\le p} R(\theta_\varepsilon,\theta_\varepsilon^j) \le \max_{j\ge N_\varepsilon} R(\theta_\varepsilon,\theta_\varepsilon^j)+\varepsilon \le \underline{\mathrm{I}}+3\varepsilon,
$$

for all $\varepsilon>0$ and $\theta>1$ and all sufficiently large s. Hence, the relation $\widetilde{R}(\theta) := \overline{\lim\limits_{s\to\infty}}\, R(\theta,s) \le \underline{\mathrm{I}}$ is valid for each $\theta>1$. Now, we obtain

$$
\mathrm{I}_0 := \sup_{\theta>1}\widetilde{R}(\theta) \le \underline{\mathrm{I}} \ \text{ and } \ \overline{\lim_{\theta\to1+0}}\,\widetilde{R}(\theta) \le \underline{\mathrm{I}}.
$$

On the other hand, $\underline{\lim\limits_{\theta\to1+0}}\,\widetilde{R}(\theta) \ge \overline{\lim\limits_{\theta\to1+0}}\, R(\theta) = \underline{\mathrm{I}}$, since $\widetilde{R}(\theta)\ge R(\theta)$. Thus,

$$
\underline{\lim_{\theta\to1+0}}\,\widetilde{R}(\theta) \ge \underline{\mathrm{I}} \ge \overline{\lim_{\theta\to1+0}}\,\widetilde{R}(\theta)
$$

and therefore the limit $\lim\limits_{\theta\to1+0}\widetilde{R}(\theta) = \underline{\mathrm{I}} \ge \mathrm{I}_0$ exists. Since the last inequality is possible only as an equality, we have the required assertion. □

Remark. The above proof essentially follows from the well-known scheme of the similar proof for general exponent, see [1, p. 110], [4, p. 67], or [5, p. 61].

Proposition 2. *For any nonzero solution x to system* (1) *the following equalities*

$$
\begin{aligned}
\iota[x] &= \sup_{\theta>0}\,\overline{\lim_{s\to+\infty}}\,\frac{1}{(\theta-1)s}\,\ln\frac{\|x(\theta s)\|}{\|x(s)\|} = \lim_{\theta\to1+0}\,\overline{\lim_{s\to+\infty}}\,\frac{1}{(\theta-1)s}\,\ln\frac{\|x(\theta s)\|}{\|x(s)\|} \\
&= \lim_{\theta\to1+0}\,\overline{\lim_{k\to\infty}}\,\frac{1}{(\theta-1)\theta^k}\,\ln\frac{\|x(\theta^{k+1})\|}{\|x(\theta^k)\|}
\end{aligned}
$$

are valid.

To prove Proposition 2, we use some theorems from [11] concerning the growth of x instead of standard estimates for the Cauchy matrix used in the proof of Proposition 1, but the rest of the proof is rather analogous to previous one.

Definition. The number

$$
\iota_\theta[x] := \overline{\lim_{s\to+\infty}}\,\frac{1}{(\theta-1)s}\,\ln\frac{\|x(\theta s)\|}{\|x(s)\|}
$$

is called the θ-uniform exponent of a nonzero solution x to system (1).

Together with original system (1), consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{7}$$

with piecewise continuous bounded perturbation matrix Q. Let $\mathfrak{R}_\sigma$ be the set of all piecewise continuous bounded perturbations Q such that

$$\lambda[Q] = \overline{\lim_{t\to+\infty}}\, t^{-1} \ln \|Q(t)\| < -\sigma, \ \ \sigma \in \mathbb{R}.$$

Put

$$\mathrm{i}_\theta(A+Q) = \sup_y \iota_\theta[y],$$

where the supremum is taken over all non-trivial solutions of system (7).

Theorem. *For any $(\alpha, \beta) \in M(A)$, there exists a number $\theta > 1$ such that*

$$\alpha = \sup \big\{ \mathrm{i}_\theta(A+Q) : \ Q \in \mathfrak{R}_\beta \big\}.$$

The proof is based on Millionshchikov's rotation method [10], [3], [5, p. 75].

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