Asymptotic Behaviour of Solutions of n-Order Differential Equations with Regularly Varying Nonlinearities

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Consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, the $\varphi_j : \Delta Y_j \to]0; +\infty[$ are continuous functions regularly varying as $y^{(j)} \to Y_j$ of order σ_j, $j = \overline{0, n-1}$, ΔY_j is a one-sided neighborhood of the point Y_j, $Y_j \in \{0, \pm\infty\}$[1].

The equation (1) is a particular case of the equation, comprehensively studied by V. M. Evtukhov and A. M. Klopot [3]

$$y^{(n)} = \sum_{k=1}^{m} \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}),$$

where $n \geq 2$, $\alpha_k \in \{-1, 1\}$ $(k = \overline{1, m})$, the $p_k : [a, \omega[\to]0, +\infty[$ $(k = \overline{1, m})$ are continuous functions, $-\infty < a < \omega \leq +\infty$, the $\varphi_{kj} : \Delta Y_j \to]0; +\infty[$ $(k = \overline{1, m},\ j = \overline{0, n-1})$ are continuous functions regularly varying as $y^{(j)} \to Y_j$ of order σ_j, ΔY_j is a one-sided neighborhood of the point Y_j, Y_j is equal to either 0 or $\pm\infty$.

From mentioned results necessary and sufficient existence conditions of the so-called $\mathcal{P}_{+\infty}(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of equation (1) can be obtained for all λ_0 $(-\infty \leq \lambda_0 \leq +\infty)$. Moreover, asymptotic representations as $t \to +\infty$ of such solutions and their derivatives of order up to $n-1$ can be established.

It follows from the definition of these solutions that

$$\lim_{t \to +\infty} y^{(j)}(t) = Y_j \in \{0, \pm\infty\} \ \ (j = \overline{0, n-1}), \quad \lim_{t \to +\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0.$$

However, the set of the monotonous solutions of equation (1), defined in some neighborhood of $+\infty$, also can have the solutions such that for each of them there exists a number $k \in \{1, \dots, n\}$ so that

$$y^{(n-k)}(t) = c + o(1) \ \ (c \neq 0) \ \text{ as } \ t \to +\infty. \tag{2}$$

When $k = 1, 2$ or the functions $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2})$ tend to positive constants, as $y^{(i)} \to Y_i$, a question on the existence of the solutions of type (2) of equation (1) can be solved without any assumption on the limits. Otherwise, we can not get asymptotic formulas of these solutions and their derivatives of order up to $n-1$ directly from equation (1).

Some results concerning existence of solutions of type (2) were obtained in corollary 8.2 of the monograph by I. T. Kiguradze and T. A. Chanturia [2, Ch. II, § 8, p. 207] for general type equations. But these results provide for considerably strict restriction on the $(n-k+1)$-st derivative of solution.

[1]For $Y_j = \pm\infty$ here and in the following all numbers in the neighborhood ΔY_j are assumed to have constant sign.

In order to receive new results with less strict restrictions on behaviour of this and following derivatives of order $\le n-1$ in case, when $k \in \{3,\dots,n\}$ and not all $\varphi_i(y^{(i)})$ $(i=\overline{n-k+1,n-2})$ tend to positive constant as $y^{(i)} \to Y_i$, let us introduce the following definition.

Definition. The solution y of the differential equation (1) is referred (for $k \in \{3,\dots,n\}$) to as a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0,+\infty[\, \subset [a,+\infty[$ and satisfies the conditions

$$\lim_{t\to+\infty} y^{(n-k)}(t) = c \;\; (c \ne 0), \quad \lim_{t\to+\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t)y^{(n)}(t)} = \lambda_0. \tag{3}$$

It is obvious that by virtue of the first relative (3) for these solutions the following representations hold

$$y^{(l-1)}(t) = \frac{ct^{n-l-k+1}}{(n-l-k+1)!}\,[1+o(1)] \;\; (l=\overline{1,n-k}) \;\text{ as }\; t\to+\infty \tag{4}$$

and $c \in \Delta Y_{n-k}$.

It readily follows from the form of equation (1) that $y^{(n)}(t)$ has constant sign in some neighborhood of $+\infty$. Then $y^{(n-l)}(t)$ $(l=\overline{1,k-1})$ are strictly monotone functions in neighborhood of $+\infty$ and by virtue of (2) can tend only to zero as $t\to+\infty$. Therefore it is necessary that

$$Y_{j-1} = 0 \;\text{ for }\; j=\overline{n-k+2,n}. \tag{5}$$

Let us introduce the numbers μ_j $(j=\overline{0,n-1})$

$$\mu_j = \begin{cases} 1 & \text{if } Y_j=+\infty, \\ & \text{or } Y_j=0, \text{ and } \Delta Y_j \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if } Y_j=-\infty, \\ & \text{or } Y_j=0 \text{ and } \Delta Y_j \text{ is a left neighborhood of the point } 0, \end{cases}$$

such that

$$\mu_j\mu_{j+1} > 0 \;\text{ for }\; j=\overline{0,n-k-1}, \quad \mu_j\mu_{j+1} < 0 \;\text{ for }\; j=\overline{n-k+1,n-2}. \tag{6}$$

Besides, note that in some neighborhood of $+\infty$

$$\operatorname{sign} y^{(j)}(t) = \mu_j \;\; (j=\overline{0,n-1}), \;\; \operatorname{sign} y^{(n)}(t) = \alpha. \tag{7}$$

In this case along with (6) the following inequality hold

$$\alpha\mu_{n-1} < 0. \tag{8}$$

Moreover, it follows from (4) that

$$Y_{j-1} = \begin{cases} +\infty & \text{if } \mu_{n-k} > 0, \\ -\infty & \text{if } \mu_{n-k} < 0, \end{cases} \quad \text{for }\; j=\overline{1,n-k}. \tag{9}$$

These conditions on μ_j $(j=\overline{0,n-1})$ and α are necessary for existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1).

The aim of the present paper is to obtain necessary and sufficient existence conditions of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions $(k \in \{3,\dots,n\})$ of equation (1) for $\lambda_0 \in \mathbb{R} \setminus \{0,\frac{1}{2},\dots,\frac{k-3}{k-2},1\}$, and establish asymptotic as $t\to+\infty$ formulas of their derivatives of order $\le n-1$. Moreover, the question on the quantity of studied solutions will be solved.

It is significant that by virtue of the obtained results by V. M. Evtukhov [1] studied solutions of equation (1) hold the following a priori asymptotic conditions.

Lemma 1. *Let $k \in \{3,\dots,n\}$ and $\lambda_0 \in \mathbb{R}\setminus\{0,\frac{1}{2},\dots,\frac{k-3}{k-2},1\}$. Then for each $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation* (1) *the following asymptotic as $t\to+\infty$ relations hold*

$$y^{(l-1)}(t) \sim \frac{[(\lambda_0-1)t]^{n-l}}{\prod\limits_{i=l}^{n-1} a_{0i}}\, y^{(n-1)}(t) \ \ (l=\overline{n-k+2,n-1}), \tag{10}$$

where $y:[t_0,+\infty[\,\to\mathbb{R}$ is an arbitrary $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation (1), *$a_{0i}=(n-i)\lambda_0-(n-i-1)$ $(i=\overline{1,n-1})$.*

We say that a continuous function $L:\Delta Y_0\to\,]0,+\infty[$ slowly varying as $y\to Y_0$ satisfies condition S_0 if

$$L(\mu e^{[1+o(1)]\ln|y|}) = L(y)[1+o(1)] \ \text{ as } \ y\to Y_0 \ \ (y\in\Delta Y_0),$$

where $\mu=\operatorname{sign} y$.

Condition S_0 is necessarily satisfied for functions L that have a nonzero finite limit as $y\to Y_0$, for functions of the form

$$L(y)=|\ln|y||^{\gamma_1}, \quad L(y)=|\ln|y||^{\gamma_1}\big|\ln|\ln|y||\big|^{\gamma_2},$$

where $\gamma_1,\gamma_2\neq 0$, and for many other functions.

We need the following auxiliary notations:

$$\gamma = 1-\sum_{j=n-k+1}^{n-1}\sigma_j, \quad \nu=\sum_{j=n-k+1}^{n-2}\sigma_j(n-j-1),$$

$$a_{0j}=(n-j)\lambda_0-(n-j-1) \ \ (j=\overline{1,n-1}), \quad C=\prod_{j=n-k+1}^{n-2}\left|\frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\right|^{\sigma_j},$$

$$I(t)=\varphi_{n-k}(c)M(c)\int_A^t p(\tau)\tau^{\nu}\varphi_0(\mu_0\tau^{n-k})\cdots\varphi_{n-k-1}(\mu_{n-k-1}\tau)\,d\tau,$$

where

$$A=\begin{cases} a_1 & \text{if } \displaystyle\int_{a_1}^{+\infty} p(\tau)\tau^{\nu}\varphi_0(\mu_0\tau^{n-k})\cdots\varphi_{n-k-1}(\mu_{n-k-1}\tau)\,d\tau=\pm\infty,\\ +\infty & \text{if } \displaystyle\int_{a_1}^{+\infty} p(\tau)\tau^{\nu}\varphi_0(\mu_0\tau^{n-k})\cdots\varphi_{n-k-1}(\mu_{n-k-1}\tau)\,d\tau<+\infty, \end{cases}$$

$a_1\geq a$ such that $\mu_{j-1}t^{n-k-j+1}\in\Delta Y_{j-1}$ $(j=\overline{1,n-k})$ for $t\geq a_1$,

$$M(c)=\prod_{j=1}^{n-k}\left|\frac{c}{(n-j-k+1)!}\right|^{\sigma_{j-1}}.$$

Theorem 1. *Let $\gamma\neq 0$, $k\in\{3,\dots,n\}$ and $\lambda_0\in\mathbb{R}\setminus\{0,\frac{1}{2},\dots,\frac{k-3}{k-2},1\}$. Then for existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation* (1), *it is necessary that $c\in\Delta Y_{n-k}$, along with* (5), (6), (8), (9) *inequalities*

$$\lambda_0<1, \quad a_{0j+1}>0 \ \ (j=\overline{n-k+1,n-2}) \tag{11}$$

hold and the following condition be satisfied:

$$\lim_{t\to+\infty}\frac{tI'(t)}{I(t)}=\frac{\gamma}{\lambda_0-1}\,. \tag{12}$$

Moreover, each solution of that kind admits along with (2) *and* (4) *the asymptotic representations* (10) *as* $t\to+\infty$ *and*

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{L_{n-1}(y^{(n-1)}(t))\prod\limits_{j=n-k+1}^{n-2}L_j\Big(\frac{[(\lambda_0-1)t]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}y^{(n-1)}(t)\Big)}=\alpha\mu_{n-1}\gamma CI(t)[1+o(1)].$$

Theorem 2. *Let* $\gamma\neq0$, $k\in\{3,\dots,n\}$, $\lambda_0\in\mathbb{R}\setminus\{0,\frac{1}{2},\dots,\frac{k-3}{k-2},1\}$ *and functions* L_j $(j=\overline{n-k+1,n-1})$ *slowly varying as* $y^{(j)}\to Y_j$ *satisfy condition* S_0. *In addition, let* $c\in\Delta Y_{n-k}$ *and conditions* (5), (6), (8), (9), (11) *and* (12) *hold. Then, in case of existence of* $\mathcal{P}^k_{+\infty}(\lambda_0)$*-solutions of equation* (1),

$$\int\limits_{a_2}^{+\infty}\tau^{k-2}\Big|I(\tau)L_{n-1}(\mu_{n-1}\tau^{\frac{1}{\lambda_0-1}})\prod_{j=n-k+1}^{n-2}L_j(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}})\Big|^{\frac{1}{\gamma}}\,d\tau<+\infty, \tag{13}$$

where $a_2\geq a_1$ *such that* $\mu_{j-1}t^{\frac{a_{0j}}{\lambda_0-1}}\in\Delta Y_{j-1}$ $(j=\overline{n-k+2,n-1})$, $\mu_{n-1}t^{\frac{1}{\lambda_0-1}}\in\Delta Y_{n-1}$ *for* $t\geq a_2$, *and each solution of that kind admits along with* (4) *the following asymptotic representations as* $t\to+\infty$

$$\begin{aligned}
y^{(n-k)}(t)&=c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)[1+o(1)],\\
y^{(l-1)}(t)&=\frac{\mu_{n-1}(\lambda_0-1)^{n-l}t^{n-l-k+2}}{\prod\limits_{i=l}^{n-1}a_{0i}}\,W'(t)[1+o(1)]\quad(l=\overline{n-k+2,n-1}),\\
y^{(n-1)}(t)&=\mu_{n-1}\,\frac{W'(t)}{t^{k-2}}\,[1+o(1)],
\end{aligned} \tag{14}$$

where

$$W(t)=\int\limits_{+\infty}^{t}\tau^{k-2}\Big|\gamma CI(\tau)L_{n-1}(\mu_{n-1}\tau^{\frac{1}{\lambda_0-1}})\prod_{j=n-k+1}^{n-2}L_j(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}})\Big|^{\frac{1}{\gamma}}\,d\tau.$$

If the inequality $\sigma_{n-1}\neq1$ *holds along with mentioned conditions, then equation* (1) *has at least one* $\mathcal{P}^k_{+\infty}(\lambda_0)$*-solution that admits such representations. Moreover, for each* $c\in\Delta Y_{n-k}$ *in case* $\lambda_0\in]-\infty,\frac{k-2}{k-1}[\setminus\{0,\frac{1}{2},\dots,\frac{k-3}{k-2}\}$ $(\lambda_0\in[\frac{k-2}{k-1};1[)$ *there exists an* $(n-k+1)$*-parameter* ($(n-k)$*-parameter, respectively*) *family of solutions with such representations if* $\sigma_{n-1}>1$, *and* $(n-k)$*-parameter* ($(n-k-1)$*-parameter*) *if* $\sigma_{n-1}<1$.

References

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