Uniqueness of a Solution and Convergence of Finite Difference Scheme for One System of Nonlinear Integro-Differential Equations

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We consider one-dimensional analog of the following system which arises in the mathematical modeling of process of an electromagnetic field penetration into a substance [11]

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\bigg[a\bigg(\int\limits_0^t |\operatorname{rot} H|^2\,d\tau\bigg)\operatorname{rot} H\bigg], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that system (1) is obtained by the reduction of the well-known Maxwell's equations to the integro-differential form [2]. There are many works devoted to the investigation of the particular cases of system (1) (see, for example, [1–10, 12–14, 16] and the references therein).

Let us consider the following magnetic field

$$H = (0, U, V),$$

where

$$U = U(x,t), \quad V = V(x,t).$$

Then from (1) we get the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial U}{\partial x}\Big], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial V}{\partial x}\Big], \tag{2}$$

where

$$S(x,t) = \int\limits_0^t \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau. \tag{3}$$

In [13], some generalization of system of type (1) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time but independent of the space coordinates, the process of penetration of the magnetic field into the material is modeled by so-called averaged integro-differential model, (2), (3) type analog of which have the following form:

$$\frac{\partial U}{\partial t} = a(S)\,\frac{\partial^2 U}{\partial x^2}\,, \quad \frac{\partial V}{\partial t} = a(S)\,\frac{\partial^2 V}{\partial x^2}\,, \tag{4}$$

where

$$S(t) = \int\limits_0^t \int\limits_0^1 \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,dx\,d\tau. \tag{5}$$

The existence of solutions of the corresponding initial-boundary value problems for the models of type (2), (3) and (4), (5) are studied in many works (see, for example, [1–5, 12–14, 16] and the references therein).

Our aim is to study the existence and uniqueness of solutions and discrete analog for the initial-boundary value problem with mixed boundary conditions for system (4), (5) in case $a(S) = (1+S)^p$, $0 < p \le 1$.

Thus, in the domain $[0,1]\times[0,\infty)$ let us consider the following initial-boundary value problem:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \Big(1+\int_0^t\int_0^1 \Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p\,\frac{\partial^2 U}{\partial x^2}\,,\\
\frac{\partial V}{\partial t} &= \Big(1+\int_0^t\int_0^1 \Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p\,\frac{\partial^2 V}{\partial x^2}\,,
\end{aligned}\tag{6}$$

$$U(0,t)=V(0,t)=0,\quad \frac{\partial U}{\partial x}\Big|_{x=1}=\frac{\partial V}{\partial x}\Big|_{x=1}=0,\ \ t\ge 0,\tag{7}$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ \ x\in[0,1],\tag{8}$$

where $0<p\le 1$; U_0 and V_0 are given functions.

The following statement takes place.

Theorem 1. *If $0<p\le 1$, U_0, $V_0\in H^2(0,1)$ and conditions of coincidence are fulfilled, then there exists unique solution (U,V) of problem (6)–(8) such that: $U,V\in L_2(0,\infty;H^2(0,1))$, $U_{xt},V_{xt}\in L_2(0,\infty;L_2(0,1))$.*

We use usual $L_2(0,1)$ and Sobolev spaces $H^2(0,1)$.

The existence part of the Theorem 1 is proved using Galerkin's modified method and compactness arguments as in [15, 18] for nonlinear parabolic equations and as it is carried out for the case of one component magnetic field in works [2–4].

As to uniqueness of a solution, we assume that there exist two different (U_1,V_1) and (U_2,V_2) solutions of problem (6)–(8) and introduce the differences $Z=U_2-U_1$ and $W=V_2-V_1$. To show that $Z=W\equiv 0$ the following identity, analogue of Hadamard formula, is mainly used:

$$\begin{aligned}
&\Big\{\Big(1+\int_0^t\int_0^1 \Big[\Big(\frac{\partial U_2}{\partial x}\Big)^2+\Big(\frac{\partial V_2}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p\,\frac{\partial U_2}{\partial x}\\
&\qquad -\Big(1+\int_0^t\int_0^1 \Big[\Big(\frac{\partial U_1}{\partial x}\Big)^2+\Big(\frac{\partial V_1}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p\,\frac{\partial U_1}{\partial x}\Big\}\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\\
&+\Big\{\Big(1+\int_0^t\int_0^1 \Big[\Big(\frac{\partial U_2}{\partial x}\Big)^2+\Big(\frac{\partial V_2}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p\,\frac{\partial V_2}{\partial x}\\
&\qquad -\Big(1+\int_0^t\int_0^1 \Big[\Big(\frac{\partial U_1}{\partial x}\Big)^2+\Big(\frac{\partial V_1}{\partial x}\Big)^2\Big]\,dx\,d\tau\Big)^p\,\frac{\partial V_1}{\partial x}\Big\}\Big(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\Big)\\
&=\int_0^1 \frac{d}{d\mu}\Big(1+\int_0^t\int_0^1 \Big\{\Big[\frac{\partial U_1}{\partial x}+\mu\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]^2+\Big[\frac{\partial V_1}{\partial x}+\mu\Big(\frac{\partial V_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]^2\Big\}\,dx\,d\tau\Big)^p
\end{aligned}$$

$$
\begin{aligned}
&\times \Big[\frac{\partial U_1}{\partial x}+\mu\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]\, d\mu\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\\
&+\int\limits_0^1 \frac{d}{d\mu}\Bigg(1+\int\limits_0^t\int\limits_0^1\Big\{\Big[\frac{\partial U_1}{\partial x}+\mu\Big(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]^2+\Big[\frac{\partial V_1}{\partial x}+\mu\Big(\frac{\partial V_2}{\partial x}-\frac{\partial U_1}{\partial x}\Big)\Big]^2\Big\}\,dx\,d\tau\Bigg)^p\\
&\times \Big[\frac{\partial V_1}{\partial x}+\mu\Big(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\Big)\Big]\, d\mu\Big(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\Big).
\end{aligned}
$$

Now, let us consider the finite difference scheme for problem (6)–(8). On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,\dots,M$; $j=0,1,\dots,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is designed by (u_i^j,v_i^j) and the exact solution to problem (6)–(8) by (U_i^j,V_i^j). We will use the following known notations [17]:

$$
r_{x,i}^j=\frac{r_{i+1}^j-r_i^j}{h}\,,\quad r_{\overline{x},i}^j=\frac{r_i^j-r_{i-1}^j}{h}\,.
$$

Introduce inner products and norms:

$$
(r^j,g^j)=h\sum_{i=1}^{M-1} r_i^j g_i^j,\quad (r^j,g^j]=h\sum_{i=1}^{M} r_i^j g_i^j,
$$
$$
\|r^j\|=(r^j,r^j)^{1/2},\quad \|r^j]|=(r^j,r^j]^{1/2}.
$$

For problem (6)–(8), let us consider the following finite difference scheme:

$$
\begin{gathered}
\frac{u_i^{j+1}-u_i^j}{\tau}-\Big(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M-1}\big[(u_{\overline{x},\ell}^k)^2+(v_{\overline{x},\ell}^k)^2\big]\Big)^p u_{\overline{x}x,i}^{j+1}=f_{1,i}^j,\\
\frac{v_i^{j+1}-v_i^j}{\tau}-\Big(1+\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M-1}\big[(u_{\overline{x},\ell}^k)^2+(v_{\overline{x},\ell}^k)^2\big]\Big)^p v_{\overline{x}x,i}^{j+1}=f_{2,i}^j,\\
i=1,2,\dots,M-1;\ \ j=0,1,\dots,N-1,
\end{gathered}\tag{9}
$$

$$
u_0^j=v_0^j=u_{\overline{x}\,M}^j=v_{\overline{x}\,M}^j=0,\ \ j=0,1,\dots,N,\tag{10}
$$

$$
u_i^0=U_{0,i},\quad v_i^0=V_{0,i},\ \ i=0,1,\dots,M.\tag{11}
$$

Multiplying equations in (9) scalarly by u_i^{j+1} and v_i^{j+1}, respectively, it is not difficult to get the inequalities:

$$
\|u^n\|^2+\sum_{j=1}^n\|u_{\overline{x}}^j]|^2\tau<C,\quad \|v^n\|^2+\sum_{j=1}^n\|v_{\overline{x}}^j]|^2\tau<C,\ \ n=1,2,\dots,N.\tag{12}
$$

Here and below C is a positive constant independent from τ and h.

The a priori estimates (12) guarantee the stability of scheme (9)–(11). Note that the uniqueness of a solution of scheme (9)–(11) can be proved too.

The main statement of this note can be stated as follows.

Theorem 2. *If problem* (6)–(8) *has a sufficiently smooth solution* $(U(x,t),V(x,t))$, *then the solution* $u^j=(u_1^j,u_2^j,\dots,u_M^j)$, $v^j=(v_1^j,v_2^j,\dots,v_M^j)$, $j=1,2,\dots,N$ *of the difference scheme* (9)–(11) *tends to the solution of the continuous problem* (6)–(8) $U^j=(U_1^j,U_2^j,\dots,U_M^j)$, $V^j=(V_1^j,V_2^j,\dots,V_M^j)$, $j=1,2,\dots,N$ *as* $\tau\to0$, $h\to0$ *and the following estimates are true:*

$$
\|u^j-U^j\|\le C(\tau+h),\quad \|v^j-V^j\|\le C(\tau+h).
$$

We have carried out numerous numerical experiments for problem (6)–(8) with different kind of right hand sides and initial-boundary conditions.

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