On Well-Posed Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations with Two Independent Variables

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In the rectangle $\Omega=[0,a]\times[0,b]$ consider the nonlinear hyperbolic equation

$$u^{(m,n)}=f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \tag{1}$$

$$l_j(u(\,\cdot\,,y))(y)=\varphi_j(y)\ \ (j=1,\dots,m),\quad h_k(u^{(m,0)}(x,\,\cdot\,))(x)=\psi_k^{(m)}(x)\ \ (k=1,\dots,n), \tag{2}$$

where

$$u^{(j,k)}=\frac{\partial^{j+k}u}{\partial x^j\partial y^k}\,,$$

$f:\Omega\times\mathbb{R}^{n+m+mn}\to\mathbb{R}$ is a continuous function, $\varphi_j\in C^n([0,b])$, $\psi_k\in C^m([0,a])$, $l_j:C^{m-1}([0,a])\to C^n([0,b])$ and $h_k:C^{n-1}[0,b]\to C([0,a])$ are bounded linear operators.

Initial-boundary value problems for linear hyperbolic equations and systems were studied in [1] and [2]. Initial-periodic problems for nonlinear hyperbolic systems were studied in [3].

$C^{m,n}(\Omega)$ is the Banach space of functions $u:\Omega\to\mathbb{R}$, having continuous partial derivatives $u^{(j,k)}$ $(j=0,\dots,m;\ k=0,\dots,n)$, with the norm

$$\|u\|_{C^{m,n}(\Omega)}=\sum_{j=0}^m\sum_{k=0}^n\|u^{(j,k)}\|_{C(\Omega)}.$$

$\widetilde{C}^{m,n}(\Omega)$ is the Banach space of functions $u:\Omega\to\mathbb{R}$, having continuous partial derivatives $u^{(j,k)}$ $(j=0,\dots,m;\ k=0,\dots,n;\ j+k<m+n)$, with the norm

$$\|u\|_{\widetilde{C}^{m,n}(\Omega)}=\sum_{k=0}^{n-1}\|u^{(m,k)}\|_{C(\Omega)}+\sum_{j=0}^{m-1}\sum_{k=0}^n\|u^{(j,k)}\|_{C(\Omega)}.$$

If $z\in\widetilde{C}^{m,n}(\Omega)$ and $r>0$, then

$$\widetilde{\mathcal{B}}^{m,n}(z;r)=\big\{\zeta\in\widetilde{C}^{m,n}(\Omega):\ \|\zeta-z\|_{\widetilde{C}^{m,n}}\le r\big\}.$$

Let $\mathbf{v}=(v_0,\dots,v_{n-1})$, $\mathbf{w}=(w_0,\dots,w_{m-1})$ and $\mathbf{z}=(z_{m-1\,n-1},\dots,z_{0\,0})$. For a function $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ that is is continuously differentiable with respect to $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$, set:

$$f_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})=\frac{\partial f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})}{\partial v_k}\ \ (k=0,\dots,n-1),$$

$$f_{jn}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})=\frac{\partial f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})}{\partial w_j}\ \ (j=0,\dots,m-1),$$

$$f_{jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})=\frac{\partial f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})}{\partial z_{j\,k}}\ \ (j=0,\dots,m-1;\ \ k=0,\dots,n-1),$$

$$p_{jk}[u](x,y)=f_{jk}\Big(x,y,u^{(m,0)}(x,y),\dots,u^{(m,n-1)}(x,y),u^{(0,n)}(x,y),\dots,u^{(m-1,n)}(x,y),$$
$$u^{(m-1,n-1)}(x,y),\dots,u(x,y)\Big)\ \ (j=0,\dots,m;\ \ k=0,\dots,n;\ \ j+k<m+n).$$

Definition 1. Let the function $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ be continuously differentiable with respect to the phase variables $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$. We say that problem (1), (2) to is *strongly* (u_0,r)*-well-posed*, if:

(I) it has a solution $u_0(x,y)$;

(II) in the neighborhood $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$ u_0 is the unique solution;

(III) there exists $\varepsilon_0>0$, $\delta_0>0$ and $M_0>0$ such that for any $\delta\in(0,\delta_0)$, $\widetilde{\varphi}_j\in C^n([0,b])$, $\widetilde{\psi}_k\in C^m([0,a])$, and $\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ satisfying the inequalities

$$\|\varphi_j-\widetilde{\varphi}_j\|_{C^n([0,b])}<\delta \ \ (j=1,\dots,m), \quad \|\psi_k-\widetilde{\psi}_k\|_{C^m([0,a])}<\delta \ \ (k=1,\dots,n),$$
$$\big\|f_{\mathbf{v}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})-\widetilde{f}_{\mathbf{v}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\big\|+\big\|f_{\mathbf{w}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})-\widetilde{f}_{\mathbf{w}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\big\|<\varepsilon_0,$$
$$\big|f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})-\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\big|<\delta$$

in the neighborhood $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$ the problem

$$u^{(m,n)}=\widetilde{f}\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \tag{$\widetilde{1}$}$$
$$l_j(u(\,\cdot\,,y))(y)=\widetilde{\varphi}_j(y) \ \ (j=1,\dots,m), \quad h_k(u^{(m,0)}(x,\,\cdot\,))(x)=\widetilde{\psi}_k^{(m)}(x) \ \ (k=1,\dots,n) \tag{$\widetilde{2}$}$$

has a unique solution $\widetilde{u}$ and

$$\|u-\widetilde{u}\|_{C^{m,n}(\Omega)}<M_0\delta.$$

Following [4] introduce the definition.

Definition 2. Problem (1), (2) is called strongly well-posed if it is strongly (u_0,r)-well-posed for every $r>0$.

First consider the linear case, i.e., the equation

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(x,y)u^{(m,k)}+\sum_{j=0}^{m-1}p_{jn}(x,y)u^{(j,n)}+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,y)u^{(j,k)}+q(x,y). \tag{3}$$

Theorem 1. *The linear problem* (3), (2) *is strongly well-posed if and only if:*

(i) *the problem*

$$\zeta^{(n)}=\sum_{i=0}^{n-1}p_{mi}(x,y)\zeta^{(i)}; \quad h_k(\zeta)(x)=0 \ \ (k=1,\dots,n) \tag{4}$$

has only the trivial solution for every $x\in[0,a]$*;*

(ii)

$$\xi^{(m)}=\sum_{i=0}^{m-1}p_{in}(x,y)\xi^{(i)}; \quad l_j(\xi)(x)=0 \ \ (j=1,\dots,m) \tag{5}$$

has only the trivial solution for every $y\in[0,b]$*;*

(iii) *the homogeneous problem*

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(x,y)u^{(m,k)}+\sum_{j=0}^{m-1}p_{jn}(x,y)u^{(j,n)}+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,y)u^{(j,k)}, \tag{3_0}$$
$$l_j(u(\,\cdot\,,y))(y)=0 \ \ (j=1,\dots,m), \quad h_k(u^{(m,0)}(x,\,\cdot\,))(x)=0 \ \ (k=1,\dots,n) \tag{2_0}$$

has only the trivial solution.

Theorem 2. *Let the function f be continuously differentiable with respect to the phase variables $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$, and let problem* (1), (2) *be strongly (u_0, r)-well-posed for some $r > 0$. Then problem* $(3_0), (2_0)$ *is strongly well-posed, where*

$$p_{jk}(x,y) = p_{jk}[u_0](x,y) \ \ (j = 0,\dots,m; \ \ k = 0,\dots,n).$$

Theorem 3. *Let the function f be continuously differentiable with respect to the phase variables $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$, and let there exist functions $P_{ijk} \in C(\Omega)$ such that:*

(A_0)

$$P_{1jk}(x,y) \le f_{jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \le P_{2jk}(x,y) \ \ \text{for} \ \ (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}$$
$$(j = 0,\dots,m; \ \ k = 0,\dots,n; \ \ j+k < m+n);$$

(A_1) *for every $x \in [0,a]$ and arbitrary measurable functions $p_{mk} : \Omega \to \mathbb{R}$ satisfying the inequalities*

$$P_{1mk}(x,y) \le p_{mk}(x,y) \le P_{2mk}(x,y) \ \ \text{for} \ \ (x,y) \in \Omega \ \ (k = 0,\dots,n-1), \tag{6}$$

problem (4) *has only the trivial solution;*

(A_2) *for every $y \in [0,b]$ and arbitrary measurable functions $p_{jn} : \Omega \to \mathbb{R}$ satisfying the inequalities*

$$P_{1jn}(x,y) \le p_{jn}(x,y) \le P_{2jn}(x,y) \ \ \text{for} \ \ (x,y) \in \Omega \ \ (j = 0,\dots,m-1), \tag{7}$$

problem (5) *has only the trivial solution;*

(A_3) *for arbitrary measurable functions $p_{jk} : \Omega \to \mathbb{R}$ satisfying the inequalities*

$$P_{1jk}(x,y) \le p_{jk}(x,y) \le P_{2jk}(x,y) \ \ \text{for} \ \ (x,y) \in \Omega \ \ (j=0,\dots,m, \ k=0,\dots,n; \ j+k<m+n), \tag{8}$$

problem $(3_0), (2_0)$ *has only the trivial solution.*

Then problem (1), (2) *is strongly well-posed.*

Consider the "perturbed" equation

$$\begin{aligned} u^{(m,n)} &= f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big) \\ &\quad + q(x,y,u^{(m-1,n-1)},\dots,u). \end{aligned} \tag{1_q}$$

Theorem 4. *Let the function f satisfy all of the conditions of Theorem* 3, *and $q(x,y,\mathbf{z})$ be an arbitrary continuous function such that*

$$\lim_{\|\mathbf{z}\| \to +\infty} \frac{|q(x,y,\mathbf{z})|}{\|\mathbf{z}\|} = 0 \tag{9}$$

uniformly on Ω. Then problem $(1_q), (2)$ *has at least one solution.*

Corollary 1. *Let problem* $(3_0), (2_0)$ *be well-posed, and $q(x,y,\mathbf{z})$ be an arbitrary continuous function satisfying condition* (9) *uniformly on Ω. Then the equation*

$$\begin{aligned} u^{(m,n)} &= \sum_{k=0}^{n-1} p_{mk}(x,y) u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y) u^{(j,n)} + \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} p_{jk}(x,y) u^{(j,k)} \\ &\quad + q(x,y,u^{(m-1,n-1)},\dots,u) \end{aligned}$$

has at least one solution satisfying conditions (2).

The initial-boundary conditions

$$u^{(j-1,0)}(0,y)=\varphi_j(y)\ \ (j=1,\dots,m),\quad h_k(u^{(m,0)}(x,\cdot))(x)=\psi_k^{(m)}(x)\ \ (k=1,\dots,n) \tag{10}$$

are the particular case of (2).

Theorem 5. *Let the function f be continuously differentiable with respect to the phase variables $\mathbf{v}$ and $\mathbf{w}$, and let there exist a constant M and functions P_{1mk}, $P_{2mk}\in C(\Omega)$ satisfying conditions (A_1) of Theorem 3, such that*

$$\begin{gathered}P_{1mk}(x,y)\le f_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\le P_{2mk}(x,y)\\ \textit{for}\ \ (x,y,\mathbf{v},\mathbf{w},\mathbf{z})\in\Omega\times\mathbb{R}^{n+m+mn}\ \ (k=0,\dots,n-1),\\ \big|f(x,y,\mathbf{0},\mathbf{w},\mathbf{z})\big|\le M\big(1+\|\mathbf{w}\|+\|\mathbf{z}\|\big).\end{gathered}$$

Then problem (1), (10) *is solvable. Moreover, if f is locally Lipschitz continuous with respect to* $\mathbf{z}$, *then problem is uniquely solvable.*

Remark 1. In Theorems 3–5 continuous differentiability of the function $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ with respect to $\mathbf{v}$ and $\mathbf{w}$ can be replaced by Lipschitz continuity, although that will make the formulation of the theorems more cumbersome. However, without Lipschitz continuity problem (1), (2) may not have a classical solution at all.

Indeed, in the rectangle $[0,1]\times[0,2]$ consider the characteristic value problem

$$\begin{gathered}u_{xy}=\frac{3}{2}\,u_y^{\frac{1}{3}},\\ u(0,y)=\frac{1}{2}\,(y-1)^2\ \ \text{for}\ \ y\in[0,2],\quad u_x(x,0)=0\ \ \text{for}\ \ x\in[0,1].\end{gathered}$$

It has a unique *absolutely continuous* solution

$$u(x,y)=\frac{1}{2}+\int\limits_0^y \operatorname{sgn}(t-1)(x+|t-1|)^{\frac{3}{2}}\,dt,$$

which is not a classical solution because $u_y(x,y)=\operatorname{sgn}(y-1)(x+|y-1|)^{\frac{3}{2}}$ is discontinuous along the line $y=1$.

Remark 2. In Theorem 5 condition (A_1) cannot be weakened. Indeed, in the rectangle $[0,2\pi]\times[0,1]$ consider the initial-periodic problem

$$u_{xy}=3p(u^2)u_x-\cos x, \tag{11}$$

$$u(0,y)=0\ \ \text{for}\ \ y\in[0,1],\quad u_x(x,0)=u_x(x,1)\ \ \text{for}\ \ x\in[0,2\pi], \tag{12}$$

where $p\in C^\infty(\mathbb{R})$, $p(z)z>0$ for $z\neq0$ and

$$p(z)=\begin{cases}z & \text{if}\ \ |z|<2,\\ 3\operatorname{sgn}z & \text{if}\ \ |z|>3.\end{cases}$$

Although the righthand side of the equation is smooth, problem (11), (12) has a unique *absolutely continuous* but not continuously differentiable solution $u(x)=\sin^{\frac{1}{3}}x$.

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