Multi Dimensional Boundary Value Problems for Linear Hyperbolic Equations of Higher Order

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Let $m_1, \dots, m_n$ be positive integers. In the n-dimensional box $\Omega = [0, \omega_1] \times \cdots \times [0, \omega_n]$ for the linear hyperbolic equation

$$u^{(\mathbf{m})} = \sum_{\alpha < \mathbf{m}} p_\alpha(\mathbf{x}) u^{(\alpha)} + q(\mathbf{x}) \tag{1}$$

consider the boundary conditions

$$h_{ik}\big(u^{(\mathbf{m}_{1 \cdots i-1})}(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big)(\widehat{\mathbf{x}}_i) \\ = \varphi_{ik}^{(\mathbf{m}_{1, \dots, i-1})}(\widehat{\mathbf{x}}_i) \ \text{ for } \ \widehat{\mathbf{x}}_i \in \Omega_i \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n). \tag{2}$$

Here $\mathbf{x} = (x_1, \dots, x_n)$, $\widehat{\mathbf{x}}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $\Omega_i = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n]$, $\mathbf{m} = (m_1, \dots, m_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\mathbf{m}_{1 \cdots k} = (m_1, \dots, m_k, 0, \dots, 0)$ ($\mathbf{m}_{1 \cdots k} = (0, \dots, 0)$ if $k = 0$), $\widehat{\mathbf{m}}_i = \mathbf{m} - \mathbf{m}_i$ and $\mathbf{m}_i = (0, \dots, m_i, \dots, 0)$ are multi-indices,

$$u^{(\alpha)}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,,$$

$p_\alpha \in C(\Omega)$ $(\alpha < \mathbf{m})$, $q \in C(\Omega)$, $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \dots, m_i; \ i = 1, \dots, n)$, and $h_{ik} : C^{m_i - 1}([0, \omega_i]) \to C^{\widehat{\mathbf{m}}_{i+1 \cdots n}}(\Omega_i)$ $(k = 1, \dots, m_i; \ i = 1, \dots, n)$ are bounded linear operators.

Two-dimensional initial-boundary value problems were studied in [1–3].

By a solution of problem (1), (2) we understand a classical solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (1) and boundary conditions (2).

Along with problem (1), (2) consider its corresponding homogeneous problem

$$u^{(\mathbf{m})} = \sum_{\alpha < \mathbf{m}} p_\alpha(\mathbf{x}) u^{(\alpha)}, \tag{1_0}$$

$$h_{ik}\big(u^{(\mathbf{m}_{1 \cdots i-1})}(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big)(\widehat{\mathbf{x}}_i) \\ = 0 \ \text{ for } \ \widehat{\mathbf{x}}_i \in \Omega_i \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n). \tag{2_0}$$

Remark 1. Even if $h_{ik} : C^{m_i - 1}([0, \omega_i]) \to \mathbb{R}$ are bounded linear functionals, conditions (2) are not equivalent to the conditions

$$h_{ik}\big(u(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big) = \varphi_{ik}(\widehat{\mathbf{x}}_i) \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n),$$

since the latter require the additional consistency conditions

$$h_{ik}(\varphi_{jl}) = h_{jl}(\varphi_{ik}) \ \ (k = 1, \dots, m_i; \ \ l = 1, \dots, m_j; \ \ i, j = 1, \dots, n).$$

However, the homogeneous conditions (2_0) are equivalent to the homogeneous conditions

$$h_{ik}\big(u(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big) = 0 \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n).$$

We make use of following notations and definitions.

$\operatorname{supp}\alpha = \{i \mid \ \alpha_i > 0\}$, $\|\alpha\| = |\alpha_1| + \cdots + |\alpha_n|$.

$\alpha = (\alpha_1, \dots, \alpha_n) < \beta = (\beta_1, \dots, \beta_n) \iff \alpha_i \le \beta_i$ $(i = 1, \dots, n)$ and $\alpha \ne \beta$.

$\alpha = (\alpha_1, \dots, \alpha_n) \le \beta = (\beta_1, \dots, \beta_n) \iff \alpha < \beta$, or $\alpha = \beta$.

$\mathbf{m}_{i_1\cdots i_k} = (\alpha_1, \dots, \alpha_n)$, where $\alpha_{i_j} = m_{i_j}$ $(j = 1, \dots, k)$ and $\alpha_j = 0$ if $j \notin \{i_1, \dots, i_k\}$.

$\widehat{\alpha} = \mathbf{m} - \alpha$, $\widehat{\mathbf{m}}_{i_1\cdots i_k} = \mathbf{m} - \mathbf{m}_{i_1\cdots i_k}$.

$\mathbf{x}_{i_1\cdots i_l} = (x_{i_1}, \dots, x_{i_l})$, $\Omega_{i_1\cdots i_l} = [0, \omega_{i1}] \times \cdots \times [0, \omega_{i_l}]$.

$\widehat{\mathbf{x}}_{i_1\cdots i_l} = (x_{j_1}, \dots, x_{j_{n-l}})$, $\widehat{\Omega}_{i_1\cdots i_l} = [0, \omega_{j_1}] \times \cdots \times [0, \omega_{i_{n-l}}]$, where $j_1 < j_2 < \cdots < j_{n-l}$, and $\{j_1, \dots, j_{n-l}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_l\}$.

$C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\alpha)}$, $\alpha \le \mathbf{m}$, with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\alpha \le \mathbf{m}} \|u^{(\alpha)}\|_{C(\Omega)}.$$

Definition 1. Problem (1), (2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ and $q \in C(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M\Big(\sum_{i=1}^{n}\sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_\mathbf{i}}(\Omega_i)} + \|q\|_{C(\Omega)}\Big), \tag{3}$$

where M is a positive constant independent of q and φ_{ik} $(k = 1, \dots, m_i;\ i = 1, \dots, n)$.

In the domain $\Omega_{i_1\cdots i_l}$ consider the homogeneous boundary value problem depending on the parameter $\widehat{\mathbf{x}}_{i_1\cdots i_l} \in \Omega_{i_1\cdots i_l}$

$$v^{(\mathbf{m}_{i_1\cdots i_l})} = \sum_{\alpha < \mathbf{m}_{i_1\cdots i_l}} p_{\widehat{\mathbf{m}}_{i_1\cdots i_l}+\alpha}(\mathbf{x}) v^{(\alpha)}, \tag{$1_{i_1\cdots i_l}$}$$

$$h_{i_j k}\big(v^{(\mathbf{m}_{i_1\cdots i_{j-1}})}(x_1, \dots, x_{i_{j-1}}, \bullet, x_{i_{j+1}}, \dots, x_n)\big)(\widehat{\mathbf{x}}_{i_j}) \\ = 0 \ \text{ for } \ \widehat{\mathbf{x}}_{i_j} \in \Omega_{i_j} \ \ (k = 1, \dots, m_{i_j};\ \ j = 1, \dots, l). \tag{$2_{i_1\cdots i_l}$}$$

Definition 2. Problem $(1_{i_1\cdots i_l}), (2_{i_1\cdots i_l})$ is called an *associated problem of level l.*

Associated problems of level $n - 1$ can be written in the relatively simpler form

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\alpha < \widehat{\mathbf{m}}_j} p_{\mathbf{m}_j+\alpha}(\mathbf{x}) v^{(\alpha)}, \tag{1_j}$$

$$h_{ik}\big(u^{(\mathbf{m}_{1\cdots i-1})}(x_1, \dots, x_{i-1}, \bullet, x_{i+1}, \dots, x_n)\big)(\widehat{\mathbf{x}}_i) = 0 \ \text{ for } \ \widehat{\mathbf{x}}_i \in \Omega_i \ \ (k = 1, \dots, m_i,\ i \ne j). \tag{2_j}$$

Associated problems of level $n - 1$ play a principal role in well-posedness of problem (1), (2).

Theorem 1. *Problem* (1), (2) *has Fredholm property if and only if each associated homogeneous problem* $(1_{i_1\cdots i_l}), (2_{i_1\cdots i_l})$ *has only the trivial solution for every* $\widehat{\mathbf{x}}_{i_1\cdots i_l} \in \Omega_{i_1\cdots i_l}$.

Theorem 2. *Problem* $(1),(2)$ *is well-posed if and only if problem* $(1_0),(2_0)$ *has only a trivial solution, and each associated homogeneous problem* $(1_{i_1\cdots i_l}),(2_{i_1\cdots i_l})$ *has only the trivial solution for every* $\widehat{\mathbf{x}}_{i_1\cdots i_l}\in\Omega_{i_1\cdots i_l}$.

Theorem 2′. *Problem* $(1),(2)$ *is well-posed if and only if problem* $(1_0),(2_0)$ *has only a trivial solution, and each associated homogeneous problem* $(1_j),(2_j)$ *of the level* $n-1$ *is well-posed for every* $x_j\in[0,\omega_j]$ $(j=1,\dots,n)$.

In case where the coefficients p_α are smooth functions, estimate (3) is not the most precise estimate for a solution of problem $(1),(2)$. Consider the equation

$$u^{(\mathbf{m})}=\sum_{\alpha<\mathbf{m}}p_\alpha(\mathbf{x})u^{(\alpha)}+q^{(\beta)}(\mathbf{x}). \tag{1_β}$$

Theorem 3. *Let problem* $(1),(2)$ *be well posed,* $p_\alpha\in C^{\mathbf{m}}(\Omega)$ $(\alpha<\mathbf{m})$, $\beta\le\mathbf{m}$ *and* $q\in C^\beta(\Omega)$. *Then the solution* u *of the problem* $(1_\beta),(2)$ *admits the estimate*

$$\|u\|_{C(\Omega)}\le M\Big(\sum_{i=1}^n\sum_{k=1}^{m_i}\|\varphi_{ik}\|_{C(\Omega_i)}+\|q\|_{C(\Omega)}\Big), \tag{4}$$

where M *is a positive constant independent of* q *and* φ_{ik} $(k=1,\dots,m_i;\ i=1,\dots,n)$.

Now consider the following particular cases of conditions (2):

(I) Characteristic value problem:

$$\begin{aligned}u^{(m_1,\dots,m_{i-1},k,0,\dots,0)}&(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)(\widehat{\mathbf{x}}_i)\\&=\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n).\end{aligned} \tag{5}$$

(II) Initial-Boundary value problems with $n-1$ initial conditions:

$$\begin{gathered}h_{1k}\big(u(\bullet,x_2,\dots,x_n)\big)(\widehat{\mathbf{x}}_1)=\varphi_{1k}(\widehat{\mathbf{x}}_1),\\ u^{(m_1,\dots,m_{i-1},k,0,\dots,0)}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)(\widehat{\mathbf{x}}_i)\\ =\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ 2=1,\dots,n).\end{gathered} \tag{6}$$

(III) Initial-Boundary value problems with $n-l$ initial conditions:

$$\begin{aligned}h_{ik}\big(u^{(\mathbf{m}_{1\cdots i-1})}&(x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)\big)(\widehat{\mathbf{x}}_i)\\&=\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=1,\dots,l),\\ u^{(m_1,\dots,m_{i-1},k,0,\dots,0)}&(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)(\widehat{\mathbf{x}}_i)\\&=\varphi_{ik}^{(\mathbf{m}_{1,\dots,i-1})}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=l+1,\dots,n).\end{aligned} \tag{7}$$

Corollary 1. *Then problem* $(1),(5)$ *is well-posed.*

Corollary 2. *Problem* $(1),(6)$ *is well-posed if and only if the problem*

$$\begin{gathered}z^{(m_1)}=\sum_{k=0}^{m_1-1}p_{(k,m_2,\dots,m_n)}(\mathbf{x})z^{(k)},\\ h_1(z)(x_2,\dots,x_n)=0\end{gathered}$$

has only the trivial solution for every $(x_2,\dots,x_n)\in[0,\omega_2]\times\cdots\times[0,\omega_n]$.

Corollary 3. *Problem* (1), (7) *is well-posed if and only if the problem*

$$v^{(m_1,\dots,m_l)} = \sum_{\alpha<(m_1,\dots,m_l)} p_{\alpha+(m_{l+1},\dots,m_n)}(\mathbf{x})w^{(\alpha)},$$
$$h_1\big(w(\bullet, x_2,\dots,x_l)\big)(\widehat{\mathbf{x}}_1)=0,\dots,h_l\big(w^{(m_1,\dots,m_{l-1},0)}(x_1,\dots,x_{l-1},\bullet)\big)(\widehat{\mathbf{x}}_l)=0$$

is well-posed for every $(x_{l+1},\dots,x_n)\in[0,\omega_{l+1}]\times\cdots\times[0,\omega_n]$.

Consider the particular case of equation (1)

$$u^{(\mathbf{2},\dots,\mathbf{2})} = \sum_{\alpha\in\mathcal{E}} p_\alpha(\mathbf{x}_\alpha)u^{(\alpha)} + q(\mathbf{x}), \tag{8}$$

where

$$\mathcal{E} = \Big\{(\alpha_1,\dots,\alpha_n)<(2,\dots,2) \mid \ \alpha_k=0, \ \text{ or } \ \alpha_k=2 \ \ (k=1,\dots,n)\Big\},$$

and

$$\mathbf{x}_\alpha = (x_{i_1},\dots,x_{i_k}), \quad \{i_1,\dots,i_k\}=\operatorname{supp}\widehat{\alpha}.$$

For equation (8) consider the Dirichlet and periodic boundary conditions:

$$\begin{gathered} u(0,x_2,\dots,x_n)=0, \quad u(\omega_1,x_2,\dots,x_n)=0,\\ \vdots \\ u(x_1,\dots,x_{n-1},0)=0, \quad u(x_1,\dots,x_{n-1},\omega_n)=0, \end{gathered} \tag{9}$$

and

$$\begin{gathered} u^{(i,0,\dots,0)}(0,x_2,\dots,x_n)=u^{(i,0,\dots,0)}(\omega_1,x_2,\dots,x_n) \ \ (i=0,1)\\ \vdots \\ u^{(0,\dots,0,i)}(x_1,\dots,x_{n-1},0)=u^{(0,\dots,0,i)}(x_1,\dots,x_{n-1},\omega_n)=0 \ \ (i=0,1). \end{gathered} \tag{10}$$

Corollary 4. *Let*

$$(-1)^{n+\frac{\|\alpha\|}{2}}\,p_\alpha(\mathbf{x}_\alpha)\le 0 \ \ \textit{for} \ \ \alpha\in\mathcal{E}. \tag{11}$$

Then problem (8), (9) *is well-posed.*

Corollary 5. *Let*

$$(-1)^{n+\frac{\|\alpha\|}{2}}\,p_\alpha(\mathbf{x}_\alpha)< 0 \ \ \textit{for} \ \ \alpha\in\mathcal{E}. \tag{12}$$

Then problem (8), (10) *is well-posed.*

Remark 2. In Corollary 5 strict inequality (12) cannot be replaced by the non-strict inequality (11). Indeed, consider the equation

$$u^{(2,\dots,2)} = (-1)^{n-1}\sum_{i=1}^n u_{x_ix_i} + (-1)^n u + q(x_1,\dots,x_{n-1}). \tag{13}$$

Equation (13) satisfies conditions (11) but does not satisfy (12). For problem (13), (10), all associate problems of level $n-1$ have only trivial solutions. However, none of them is well-posed, because all associate problems of level less than $n-1$ have nontrivial solutions. Let us show ill-posedness of problem (13), (10) directly, without applying Theorem 2 (ill-posedness of problem (13), (10) follows immediately from Theorem 2).

Indeed, assume that problem (13), (10) has a solution u. One can easily verify that u is a unique solution of problem (13), (10), and thus is independent of x_n. Therefore, u satisfies the equation

$$\sum_{i=1}^{n-1} u_{x_i x_i} - u = q(x_1, \dots, x_{n-1}). \tag{14}$$

From the theory of elliptic equations it is well-known, that if $q \in C(\widehat{\Omega}_n)$, then, generally speaking, u is not a classical solution, i.e., it does not belong $C^2(\widehat{\Omega}_n)$, and thus does not belong to $C^{2,\dots,2}(\widehat{\Omega}_n)$.

References

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