On Proper Oscillatory Solutions of Higher Order Emden–Fowler Type Differential Systems

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On the interval $\mathbb{R}_+ = [0, +\infty[$, we consider the differential system

$$u_1^{(n_1)} = p_1(t)|u_2|^{\lambda_1}\operatorname{sgn}(u_2), \quad u_2^{(n_2)} = p_2(t)|u_1|^{\lambda_2}\operatorname{sgn}(u_1), \tag{1}$$

where

$$n_1 + n_2 \ \text{ is even}, \ \ \lambda_1 > 0, \ \ \lambda_1\lambda_2 > 1,$$

and $p_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions such that

$$p_1(t) \geq 0, \ \ p_2(t) \leq 0 \ \text{ for } \ t \in \mathbb{R}_+.$$

If $n_1 = 1$, $n_2 = n - 1$, $\lambda_1 = 1$, $\lambda_2 = \lambda$, $p_1(t) \equiv 1$ and $p_2(t) \equiv p(t)$, then system (1) is equivalent to the Emden–Fowler type differential equation

$$u^{(n)} = p(t)|u|^{\lambda}\operatorname{sgn}(u).$$

Therefore this system may naturaly be called as Emden–Fowler type differential system.

A nontrivial solution (u_1, u_2) of system (1) defined on some infinite interval $[t_0, +\infty[\subset \mathbb{R}_+$ is said to be **proper**.

A proper solution (u_1, u_2) of (1) is said to be **oscillatory** if its components u_1 and u_2 change sign in any neighbourhood of $+\infty$.

We have established the necessary and sufficient conditions for the oscillation of all proper solutions of system (1) and also the conditions guaranteeing the existence of a multiparametric family of proper oscillatory solutions of that system.

Such results were known earlier only in the cases where $n_1 = n_2 = 1$ or $p_1(t) \equiv 1$ and $\lambda_1 = 1$ (see [1, 2] and the references therein).

Theorem 1. *If the conditions*

$$\int\limits_0^{+\infty} p_1(t)\,dt = +\infty, \tag{2}$$

$$\int\limits_0^{+\infty} t^{n_2-1}\bigg[\int\limits_0^t (t-s)^{n_1-1}\Big(\frac{s}{t}\Big)^{(n_2-1)\lambda_1} p_1(s)\,ds\bigg]^{\lambda_2} p_2(t)\,dt = -\infty, \tag{3}$$

$$\lim_{x\to+\infty}\int\limits_0^x t^{n_1-1}\bigg[\int\limits_t^x (s-t)^{n_2-1}|p_2(s)|\,ds\bigg]^{\lambda_1} p_1(t)\,dt = +\infty \tag{4}$$

are fulfilled, then every proper solution of system (1) *is oscillatory.*

If

$$\liminf_{t\to+\infty} \frac{\int\limits_0^t (t-s)^{n_1-1} s^{(n_2-1)\lambda_1} p_1(s)\,ds}{t^{(n_2-1)\lambda_1} \int\limits_0^t (t-s)^{n_1-1} p_1(s)\,ds} > 0, \tag{5}$$

then (3) *takes the form*

$$\int\limits_0^{+\infty} t^{n_2-1} \bigg[\int\limits_0^t (t-s)^{n_1-1} p_1(s)\,ds \bigg]^{\lambda_2} p_2(t)\,dt = -\infty. \tag{6}$$

Theorem 2. *Let conditions* (2) *and* (5) *be fulfilled. Then for the oscillation of all proper solutions of system* (1), *it is necessary and sufficient that equalities* (4) *and* (6) *be satisfied.*

Corollary 1. *Let there exist numbers $t_0 > 0$, $r_i > 0$ $(i = 1, 2)$, $\mu_1 \le 1$ and μ_2 such that*

$$r_1 \le t^{\mu_1} p_1(t) \le r_2, \quad r_1 \le -t^{\mu_2} p_2(t) \le r_2 \ \ \textit{for} \ \ t \ge t_0. \tag{7}$$

Then for the oscillation of all proper solutions of system (1), *it is necessary and sufficient that the inequality*

$$\mu_2 \le \frac{n_1 - \mu_1}{\lambda_1} + n_2 \tag{8}$$

be fulfilled.

Theorems 1 and 2 leave the question on the existence of proper solutions of system (1) open. The answer to this question gives the following theorem.

Theorem 3. *If n_1 is even and $n_2 = n_1$, then system* (1) *has n_1-parametric family of proper solutions satisfying the condition*

$$\int\limits_0^{+\infty} \Big(p_1(t)|u_2(t)|^{1+\lambda_1} + p_2(t)|u_1(t)|^{1+\lambda_2} \Big)\,dt < +\infty.$$

From Corollary 1 and Theorem 3 it follows

Corollary 2. *Let $n_2 = n_1$, n_1 be even and there exist numbers $t_0 > 0$, $r_2 > r_1 > 0$, $\mu_1 \le 1$ and μ_2 such that inequalities* (7) *and* (8) *are fulfilled. Then system* (1) *has n_1-parametric family of proper oscillatory solutions.*

References

[1] I. Kiguradze and T. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. *Springer Science & Business Media*, 2012.

[2] J. D. Mirzov, Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations. *Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica*, 14. *Masaryk University, Brno*, 2004.