On the Solvability of One Multidimensional Boundary Value Problem for a Semilinear Hyperbolic Equation

Sergo Kharibegashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbiisi State University, Tbilisi, Georgia
E-mail: `kharibegashvili@yahoo.com`

Consider the semilinear hyperbolic equation of the type

$$L_f u := \square^2 u + f(u) = F, \tag{1}$$

where $f : R \to R$ is a given continuous nonlinear function, F is a given and u is an unknown real function,

$$\square := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\,, \quad n \geq 2.$$

Let D be a convex domain in the space R^{n+1} of variables $x_1, \dots, x_n, t$ with piecewise – smooth boundary $S = \partial D$, consisting of smooth n-dimensional manifolds $S_1, S_2, \dots, S_{m_0}, S_{m_0+1}, \dots, S_m$ whose S_i, $i = 1, \dots, m_0$, are manifolds of spatial and temporal types, and $S_{m_0+1}, \dots, S_m$ are characteristic manifolds.

For the equation (1), we consider the boundary value problem: find in the domain D a solution $u = u(x_1, \dots, x_n, t)$ of that equation according to the boundary conditions:

$$u|_S = 0; \quad \frac{\partial u}{\partial \nu}\Big|_{S_i} = 0, \quad i = 1, \dots, m_0, \tag{2}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to ∂D.

Assume

$$\overset{\circ}{C}{}^k(D, \partial D) := \Big\{ u \in C^k(D) : \ u\big|_S = 0; \ \frac{\partial u}{\partial \nu}\Big|_{S_i} = 0, \ i = 1, \dots, m_0 \Big\}, \quad k \geq 2.$$

Let $u \in \overset{\circ}{C}{}^4(D, \partial D)$ be a classical solution of the problem (1), (2). Multiplying both parts of the equation (1) by an arbitrary function $\varphi \in \overset{\circ}{C}{}^2(D, \partial D)$ and integrating the obtained equality by parts over the domain D, we obtain

$$\int_D \square\, u \,\square\, \varphi \, dx\, dt + \int_D f(u)\varphi \, dx\, dt = \int_D F\varphi \, dx\, dt. \tag{3}$$

Introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\square}(D)$ as the completion with respect to the norm

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\square}(D)} = \int_D \Big[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i}\Big)^2 + (\square\, u)^2 \Big] \, dx\, dt$$

of the classical space $\overset{\circ}{C}{}^2(D, \partial D)$.

Consider the following conditions imposed on the function $f = f(u)$:

$$f \in C(R), \ \ |f(u)| \leq M_1 + M_2|u|^{\alpha}, \ \ u \in R, \tag{4}$$

where

$$0 \leq \alpha = const < \frac{n+1}{n-1}\,. \tag{5}$$

Let $F \in L_2(D)$. We take the equality (3) as a basis for our definition of the generalized solution u of the problem (1), (2): the function $u \in \overset{\circ}{W}{}^1_{2,\square}(D)$ is said to be a weak generalized solution of the problem (1), (2) if for any function $\varphi \in \overset{\circ}{W}{}^1_{2,\square}(D)$ the integral equality (3) is valid.

Theorem. *Let f be a monotone function and satisfy the conditions* (4), (5) *and* $uf(u) \geq 0 \ \forall\, u \in R$. *Then for any $F \in L_2(D)$ the problem* (1), (2) *has a unique weak generalized solution in the space* $\overset{\circ}{W}{}^1_{2,\square}(D)$.

As the examples show, if the conditions imposed on the nonlinear function f are violated, then the problem (1), (2) may not have a solution.

Acknowledgement

The work is supported by the Shota Rustavely National Science Foundation (Grant # FR/86/5–109/14).