On Some Sufficient Conditions for the ξ-Exponential Asymptotical Stability in the Lyapunov Sense of Systems of Linear Impulsive Equations

Nestan Kekelia

Sokhumi State University, Tbilisi, Georgia
E-mail: nest.kek@mail.ru

Consider the linear system of impulsive equations

$$\frac{dx}{dt} = Q(t)x + q(t) \ \text{ for } \ t \in \mathbb{R}_+, \tag{1}$$

$$x(t_j+) - x(t_j-) = G_j x(t_j-) + g_j \ \ (j = 1, 2, \dots), \tag{2}$$

where $Q \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$, $q \in L_{loc}(\mathbb{R}_+; \mathbb{R}^n)$, $G_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$, $g_j \in \mathbb{R}^n$ $(j = 1, 2, \dots)$, $t_j \in \mathbb{R}_+$ $(j = 1, 2, \dots)$, $0 < t_1 < t_2 < \cdots$, $\lim\limits_{j\to+\infty} t_j = +\infty$.

We use the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \ge 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

$\widetilde{C}([a, b], D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a, b] \to D$.

$\widetilde{C}_{loc}(I \setminus T, D)$, where $T = \{t_1, t_2, \dots\}$, is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I \setminus \{\tau_l\}_{l=1}^m$ belong to $\widetilde{C}([a, b], D)$.

$L([a, b]; D)$ is the set of all integrable matrix-functions $X : [a, b] \to D$.

$L_{loc}(I; D)$ is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from I_{t_0} belong to $L([a, b], D)$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector function $x : \mathbb{R}_+ \to \mathbb{R}^n$, $x \in \widetilde{C}_{loc}(\mathbb{R}_+ \setminus T; \mathbb{R}^n)$, satisfying the system (1) a.e on $]t_j, t_{j+1}[$, and the equality (2) at the point t_j for every $j \in \{1, 2, \dots\}$.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, $\xi \in \widetilde{C}_{loc}(\mathbb{R}_+; \mathbb{R}_+)$, be a continuous from the left nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty.$$

Definition 1. The solution x_0 of the system (1), (2) is said to be ξ-exponentially asymptotically stable if there is $\eta > 0$ such that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every solution x of the system (1), (2) satisfying the condition

$$\|x(t_0) - x_0(t_0)\| < \delta$$

for some $t_0 \in \mathbb{R}_+$, the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp\big(\eta(\xi(t) - \xi(t_0))\big) \text{ for } t \geq t_0$$

holds.

Definition 2. The system (1), (2) is said to be ξ-exponentially asymptotically stable if every its solution is ξ-exponentially asymptotically stable.

Definition 3. The pair $(Q, \{G_l\}_{l=1}^{\infty})$, where $Q \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$, is ξ-exponentially asymptotically stable if the corresponding to this pair homogeneous impulsive system

$$\frac{dx}{dt} = Q(t)x \text{ for } t \in \mathbb{R}_+,$$
$$x(t_j+) - x(t_j-) = G_j x(t_j-) \ (j = 1, 2, \dots)$$

is stable in the same sense.

Theorem. *Let $Q = (q_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G_j = (g_{jik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$ be such that the conditions*

$$1 + g_{jii} \neq 0 \ (i = 1, \dots, n; \ j = 1, 2, \dots),$$
$$r(H) < 1, \tag{3}$$

$$\sup\bigg\{(\xi(t) - \xi(\tau))^{-1}\bigg(\int_\tau^t q_{ii}(s)\,ds + \sum_{\tau\leq t_j<t} \ln|1 + g_{jii}|\bigg) : \\ t \geq \tau \geq t^*, \ \xi(t) \neq \xi(\tau); \ t, \tau \in \mathbb{R}_+ \setminus T\bigg\} < -\gamma \ (i = 1, \dots, n) \tag{4}$$

and

$$\int_{t^*}^t \exp\bigg(\gamma(\xi(t) - \xi(\tau)) + \int_\tau^t q_{ii}(s)\,ds\bigg)|q_{ik}(\tau)| \prod_{\tau\leq t_j<t} |1 + g_{jii}|\,d\tau \\ + \sum_{t^*\leq t_l<t} \exp\bigg(\gamma(\xi(t) - \xi(t_l)) + \int_{t_l}^t q_{ii}(s)\,ds\bigg)|g_{lik}| \prod_{t_l<t_j<t} |1 + g_{jii}| \leq h_{ik},$$
$$\text{for } t \in [t^*, +\infty[\setminus T \ (i \neq k; \ i, k = 1, \dots, n)$$

hold, where $\gamma > 0$, t^* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \dots, n)$, $H = (h_{ik})_{i,k=1}^n$ matrix, where $h_{ii} = 0$ $(i = 1, \dots, n)$. Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is ξ-exponentially asymptotically stable.

Corollary. *Let $Q = (q_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G_j = (g_{jik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$ $(j = 1, 2, \dots)$ be such that the conditions* (3), (4),

$$-1 < g_{jii} \leq 0 \ (i = 1, \dots, n; \ j = 1, 2, \dots),$$
$$q_{ii}(t) \leq 0 \ (i = 1, \dots, n),$$
$$|q_{ik}(t)| \leq -h_{ik} q_{ii}(t) \ (i \neq k; \ i, k = 1, \dots, n),$$
$$|g_{jik}| < -h_{ik} g_{jii}(1 + g_{jii}) \ (i \neq k; \ i, k = 1, \dots, n; \ j = 1, 2, \dots)$$

hold a.e on the interval $[t^*, +\infty[$, where $\gamma > 0$, t^* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \dots, n)$, $h_{ii} = 0$ $(i = 1, \dots, n)$, and $H = (h_{ik})_{i,k=1}^n$. Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is ξ-exponentially asymptotically stable.

The questions on the Lyapunov stability in this and other sense are investigated in [1, 3] (see, also the references therein) for linear impulsive systems, and analogous questions in [2] (see, also the references therein) for ordinary differential systems.

References

[1] Sh. Akhalaia, M. Ashordia, and N. Kekelia, On the necessary and sufficient conditions for the stability of linear generalized ordinary differential, linear impulsive and linear difference systems. *Georgian Math. J.* **16** (2009), no. 4, 597–616.

[2] I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. Linear theory. (Russian) *Metsniereba, Tbilisi*, 1997.

[3] A. M. Samoĭlenko and N. A. Perestyuk, Impulsive differential equations. *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, 14. *World Scientific Publishing Co., Inc., River Edge, NJ*, 1995.