Lyapunov Exponents of Parametric Families of Linear Differential Systems

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Consider parametric family of n-dimensional ($n \geq 2$) linear differential systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t,\mu)x, \;\; x \in \mathbb{R}^n, \;\; t \geq 0, \;\; \mu \in B, \tag{1}$$

whose solutions continuously depend on parameter $\mu \in B$, and B is metric space. Denote class of all such systems by $\mathcal{S}_n^*$. By $\mathcal{S}_n$ we denote subclass of $\mathcal{S}_n^*$ of such systems that for any $\mu \in B$ coefficient matrix $A(\,\cdot\,,\mu)$ is bounded over all $t \geq 0$. We identify family (1) and it's coefficient matrix and therefore write $A \in \mathcal{S}_n^*$ or $A \in \mathcal{S}_n$. For any $A \in \mathcal{S}_n^*$ and $\mu \in B$ by A_μ we denote differential system of family (1) with fixed parameter μ.

For any family $A \in \mathcal{S}_n^*$ let $\lambda_1(\mu) \leq \cdots \leq \lambda_n(\mu)$ be Lyapunov exponents of system A_μ. Lyapunov exponents $\lambda_i(\mu)$, $i = \overline{1,n}$, are real numbers for all families $A \in \mathcal{S}_n$, therefore we consider $\lambda_i(\,\cdot\,)$ as functions $B \to \mathbb{R}$. For families $A \in \mathcal{S}_n^*$, generally speaking, Lyapunov exponents $\lambda_i(\mu)$, $i = \overline{1,n}$ can take improper values, therefore we consider $\lambda_i(\,\cdot\,)$ as functions $B \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{-\infty, +\infty\}$.

All statements given below are true in essentially more general case of Lyapunov exponents of families of morphisms of Millionshtchikov bundles and generalized Millionshtchikov bundles. Nevertheless we use the more familiar language of Lyapunov exponents of parametric families (1).

Lyapunov exponents of families $A \in \mathcal{S}_n$ as functions $B \to \mathbb{R}$ are completely described using Baire characterization. V. M. Millionschikov [5] proved that every function $\lambda_k(\,\cdot\,) : B \to \mathbb{R}$ is a function of the second Baire class. M. I. Rakhimberdiev [7] proved that the number of Baire class in the statement above cannot be reduced. A. N. Vetokhin [8], [9] in special spaces of differential systems proved that Lyapunov exponents considered as functions of systems belong to the Baire class $({}^*, G_\delta)$. Recall that a real-valued function is referred to as a function of the class $({}^*, G_\delta)$ [1, pp. 223, 224] if for each $r \in \mathbb{R}$ the pre-image of the interval $[r, +\infty)$ under the mapping f is a G_δ-set, i.e. can be represented as a countable intersection of open sets. A complete description of Lyapunov exponents of families $A \in \mathcal{S}_n$ as functions $B \to \mathbb{R}$ was announced if [2] and presented in [3]. For any positive integer n and metric space B set $(f_1(\,\cdot\,), \dots, f_n(\,\cdot\,))$ of functions $B \to \mathbb{R}$ coincides with set of Lyapunov exponents $(\lambda_1(\,\cdot\,), \dots, \lambda_n(\,\cdot\,))$ of some family $A \in \mathcal{S}_n$ if and only if all these functions belong to the Baire class $({}^*, G_\delta)$, have upper semi-continuous minorant and satisfy inequalities $f_1(\mu) \leq \cdots \leq f_n(\mu)$ for all $\mu \in B$.

Consider the same problem of description of Lyapunov exponents of families $A \in \mathcal{S}_n^*$ as functions $B \to \overline{\mathbb{R}}$. V. M. Millionschikov [6] proved that every function $\lambda_k(\,\cdot\,) \colon B \to \overline{\mathbb{R}}$ is a function of the second Baire class. A complete solution of this problem is given by the following theorem.

Theorem 1. *For any positive integer n and metric space B set $(f_1(\,\cdot\,), \dots, f_n(\,\cdot\,))$ of functions $B \to \overline{\mathbb{R}}$ coincides with set of Lyapunov exponents $(\lambda_1(\,\cdot\,), \dots, \lambda_n(\,\cdot\,))$ of some family $A \in \mathcal{S}_n^*$ if and only if all these functions belong to the Baire class $({}^*, G_\delta)$ and satisfy inequalities $f_1(\mu) \leq \cdots \leq f_n(\mu)$ for all $\mu \in B$.*

Here for functions $B \to \overline{\mathbb{R}}$ we use the same definition of the Baire class $({}^*, G_\delta)$: function $B \to \overline{\mathbb{R}}$ is referred to as a function of the class $({}^*, G_\delta)$ if for each $r \in \overline{\mathbb{R}}$ the preimage of the segment $[r, +\infty]$ under the mapping f is a G_δ-set.

Consider family $A \in \mathcal{S}_n$. For every Lyapunov exponent $\lambda_i(\,\cdot\,)$ consider set M_i of all points $\mu \in B$ at which function $\lambda_i(\,\cdot\,)$ is upper (lower) semi-continuous. Set $(M_1, M_2, \dots, M_n)$ we call the set of upper (lower) semi-continuity of Lyapunov exponents of family A. V. M. Millionschikov [6] proved that if parameter space B is full metric space, then upper semi-continuity is Baire typical for all Lyapunov exponents i.e. for any $A \in \mathcal{S}_n$ and $i = \overline{1, n}$ the set M_i of upper semi-continuity contains dense G_δ-subset. A. N. Vetokhin showed that sets of lower semi-continuity can be empty.

Sets of upper semi-continuity and lower semi-continuity of families $A \in \mathcal{S}_n$ are completely described in [4]. In the case of Lyapunov exponents of families $A \in \mathcal{S}_n^*$ the description of upper and lower semi-continuity sets turned out to be the same. This description is given in the next theorem.

Theorem 2. *For any positive integer n and full metric space B set $(M_1, \dots, M_n)$ of subsets of space B is the set of upper semi-continuity of Lyapumov exponents of some family $A \in \mathcal{S}_n^*$ if and only if every M_i, $i = \overline{1, n}$ is dense G_δ-set, and the set of lower semi-continuity of Lyapumov exponents of some family $A \in \mathcal{S}_n^*$ if and only if every M_i, $i = \overline{1, n}$ is $F_{\sigma\delta}$-set which contains all isolated points of space B.*

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