Structure and Asymptotic Behavior of Nonoscillatory Solutions of First-order Cyclic Functional Differential Systems

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We consider first-order cyclic functional differential systems of the type

$$x'(t) + p(t)\varphi_\alpha(y(k(t))) = 0, \quad y'(t) + q(t)\varphi_\beta(x(l(t))) = 0, \tag{A}$$

under the assumption that

(a) α and β are positive constants;

(b) $p(t)$ and $q(t)$ are positive continuous functions on $[0,\infty)$;

(c) $k(t)$ and $l(t)$ are positive continuous functions on $[0,\infty)$ tending to ∞ as $t\to\infty$;

(d) $\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u = |u|^{\gamma-1}u$, $\gamma>0$, $u\in\mathbf{R}$.

Let $T>0$ be a fixed point on the real line. Define T_0 by

$$T_0 = \min\Big\{T, \inf_{t\ge T} k(t), \inf_{t\ge T} l(t)\Big\}.$$

By a *solution* of system (A) on $[T,\infty)$ we mean a vector function $(x(t), y(t))$ which is defined on $[T_0,\infty)$ and satisfies (A) for all $t\in[T,\infty)$. Such a solution is called *oscillatory* (or *nonoscillatory*) if both components of it are oscillatory (or nonoscillatory) in the usual sense. It is clear that (A) admits no oscillatory solutions, so that all nontrivial solutions of (A), if exist, are nonoscillatory.

Let $(x(t), y(t))$ be a nonoscillatory solution of (A). Since (A) implies that $x(t)$ and $y(t)$ are eventually monotone, the two cases may occur: either (Case I) $x(t)y(t)>0$ or (Case II) $x(t)y(t)<0$ for all large t. In either case the limits $x(\infty) = \lim\limits_{t\to\infty} x(t)$ and $y(\infty)=\lim\limits_{t\to\infty} y(t)$ exist in the extended real numbers.

Suppose that $x(t)y(t)>0$ for all large t. Then, $|x(t)|$ and $|y(t)|$ are eventually decreasing, and so there are the following three possibilities for the combination $(x(\infty), y(\infty))$:

I(i) $0 < |x(\infty)| < \infty$, $0 < |y(\infty)| < \infty$;

I(ii) (a) $0 < |x(\infty)| < \infty$, $|y(\infty)| = 0$, or
(b) $|x(\infty)| = 0$, $0 < |y(\infty)| < \infty$;

I(iii) $|x(\infty)| = 0$, $|y(\infty)| = 0$.

Suppose that $x(t)y(t) < 0$ for all large t. In this case $|x(t)|$ and $|y(t)|$ are eventually increasing, and there are the following three possibilities for the combination $(|x(\infty)|, |y(\infty)|)$:

II(i) $|x(\infty)| < \infty$, $|y(\infty)| < \infty$;

II(ii) (a) $|x(\infty)| < \infty$, $|y(\infty)| = \infty$, or
(b) $|x(\infty)| = \infty$, $|y(\infty)| < \infty$;

II(iii) $|x(\infty)| = \infty$, $|y(\infty)| = \infty$.

The existence of nonoscillatory solutions of the four types I(i), I(ii), II(i) and II(ii) can be completely characterized as shown in the following theorems.

Theorem 1. *System* (A) *has a solution* $(x(t), y(t))$ *such that* $x(t)y(t) > 0$ *for all large* t *and*

$$\lim_{t\to\infty} x(t) = const \neq 0, \quad \lim_{t\to\infty} y(t) = const \neq 0,$$

if and only if

$$\int_0^\infty p(t)\,dt < \infty \ \ and \ \int_0^\infty q(t)\,dt < \infty.$$

Theorem 2. *System* (A) *has a solution* $(x(t), y(t))$ *such that* $x(t)y(t) > 0$ *for all large* t *and*

$$\lim_{t\to\infty} x(t) = const \neq 0, \quad \lim_{t\to\infty} y(t) = 0,$$

if and only if

$$\int_0^\infty q(t)\,dt < \infty \ \ and \ \int_0^\infty p(t)\rho(k(t))^\alpha\,dt < \infty,$$

where

$$\rho(t) = \int_t^\infty q(s)\,ds.$$

Theorem 3. *System* (A) *has a solution* $(x(t), y(t))$ *such that* $x(t)y(t) < 0$ *for all large* t *and*

$$\lim_{t\to\infty} x(t) = const \neq 0, \quad \lim_{t\to\infty} y(t) = const \neq 0,$$

if and only if

$$\int_0^\infty p(t)\,dt < \infty \ \ and \ \int_0^\infty q(t)\,dt < \infty.$$

Theorem 4. *System* (A) *has a solution* $(x(t),y(t))$ *such that* $x(t)y(t)<0$ *for all large* t *and*

$$\lim_{t\to\infty}|x(t)|=const\neq 0,\quad \lim_{t\to\infty}|y(t)|=\infty,$$

if and only if

$$\int_0^\infty q(t)\,dt=\infty \ \ \text{and} \ \ \int_0^\infty p(t)Q(k(t))^\alpha\,dt<\infty,$$

where

$$Q(t)=\int_0^t q(s)\,ds.$$

Note that the theorems concerning the cases I(iib) and II(iib) could be formulated automatically from Theorems 2 and 4, respectively.

The solutions of types I(iii) and II(iii) seem to be extremely difficult to analyze, and for the present we have to content ourselves with seeking *regularly varying solutions* for system (A) in which $\alpha\beta<1$, $p(t)$ and $q(t)$ are regularly varying and $k(t)$ and $l(t)$ are regularly varying of index 1. By a regularly varying solution of system (A) we here mean a nonoscillatory solution $(x(t),y(t))$ of (A) such that both $|x(t)|$ and $|y(t)|$ are regularly varying in the sense of Karamata. If $|x|\in \mathrm{RV}(\rho)$ and $|y|\in\mathrm{RV}(\sigma)$, we write $(x,y)\in\mathrm{RV}(\rho,\sigma)$, and call $(x(t),y(t))$ a regularly varying solution of index (ρ,σ).

In the following theorems it is assumed that $p\in\mathrm{RV}(\lambda)$ and $q\in\mathrm{RV}(\mu)$ and they have the expressions

$$p(t)=t^\lambda L(t),\quad q(t)=t^\mu M(t),\ \ L,M\in\mathrm{SV},$$

and that $k(t)$ and $l(t)$ satisfy

$$\lim_{t\to\infty}\frac{k(t)}{t}=\gamma,\quad \lim_{t\to\infty}\frac{l(t)}{t}=\delta,$$

for some positive constants γ and δ, respectively.

First we look for regularly varying solutions of type I(iii). It is clear that $(x,y)\in\mathrm{RV}(\rho,\sigma)$ is of type I(iii) (i.e., $x(\infty)=y(\infty)=0$) if (ρ,σ) falls into one of the three cases:

(a) $\rho<0$, $\sigma<0$,

(b) $\rho=0$, $\sigma<0$, or $\rho<0$, $\sigma=0$,

(c) $\rho=\sigma=0$.

We are able to deal with the cases (a) and (b) exhaustively. Our result for the case (a) follows.

Theorem 5. *Let* $\alpha\beta<1$. *Suppose that* λ *and* μ *satisfy*

$$\lambda+1+\alpha(\mu+1)<0,\quad \beta(\lambda+1)+\mu+1<0,$$

and define ρ *and* σ *by*

$$\rho=\frac{\lambda+1+\alpha(\mu+1)}{1-\alpha\beta},\quad \sigma=\frac{\beta(\lambda+1)+\mu+1}{1-\alpha\beta}.$$

Then system (A) *possesses a nonoscillatory solution* $(x(t),y(t))$ *of type* I(iii) *which satisfies* $x(t)y(t)>0$ *for all large* t *and belongs to the class* $\mathrm{RV}(\rho,\sigma)$. *The asymptotic behavior of the components* $x(t)$ *and* $y(t)$ *are governed by the precise decay laws:*

$$|x(t)|\sim t^\rho\Big[\Big(\frac{\gamma^{\alpha\sigma}L(t)}{-\rho}\Big)\Big(\frac{\delta^{\beta\rho}M(t)}{-\sigma}\Big)^\alpha\Big]^{\frac{1}{1-\alpha\beta}},\quad |y(t)|\sim t^\sigma\Big[\Big(\frac{\gamma^{\alpha\sigma}L(t)}{-\rho}\Big)^\beta\Big(\frac{\delta^{\beta\rho}M(t)}{-\sigma}\Big)\Big]^{\frac{1}{1-\alpha\beta}},$$

as $t\to\infty$.

As for the case (b) it suffices to present the result for solutions belonging to $\mathrm{RV}(0,\sigma)$ with $\sigma<0$, from which, as is easily seen, the result for solutions in $\mathrm{RV}(\rho,0)$ with $\rho<0$ can be formulated almost automatically.

Theorem 6. *Let $\alpha\beta<1$. Suppose that λ and μ satisfy*

$$\lambda=-1-\alpha(\mu+1),\quad \mu<-1.$$

Suppose moreover that for any $a>0$

$$\int\limits_a^\infty t^{-1}L(t)M(t)^\alpha\,dt=\int\limits_a^\infty p(t)(tq(t))^\alpha\,dt<\infty.$$

Put $\sigma=\mu+1$. Then system (A) *possesses a nonoscillatory solution $(x(t),y(t))$ of type* I(iii) *which satisfies $x(t)y(t)>0$ for all large t and belongs to the class $\mathrm{RV}(0,\sigma)$. The asymptotic behavior of the components $x(t)$ and $y(t)$ are governed by the precise decay laws:*

$$|x(t)|\sim\bigg[(1-\alpha\beta)\gamma^{\alpha\sigma}\int\limits_t^\infty s^{-1}L(s)\Big(\frac{M(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{1}{1-\alpha\beta}},$$

$$|y(t)|\sim t^\sigma\frac{M(t)}{-\sigma}\bigg[(1-\alpha\beta)\gamma^{\alpha\sigma}\int\limits_t^\infty s^{-1}L(s)\Big(\frac{M(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{\beta}{1-\alpha\beta}},$$

as $t\to\infty$.

In order to handle solutions of type II(iii) of (A) we note that if $(x(t),y(t))$ is a solution of (A) of that type, then $(-x(t),y(t))$ and $(x(t),-y(t))$ are solutions of the "dual" system

$$X'(t)-p(t)\varphi_\alpha(Y(k(t)))=0,\quad Y'(t)-q(t)\varphi_\beta(X(l(t)))=0, \tag{B}$$

satisfying $X(t)Y(t)>0$ for all large t and $|X(\infty)|=|Y(\infty)|=\infty$. Then the desired results for the cases (a) and (b) of II(iii) could easily be obtained from Theorems 3.1 and 3.2 established for (B) in the paper [1]. Their formulations may be omitted.

Some of the above-mentioned results for system (A) seem to be new even (A) is reduced to the ordinary differential system

$$x'+p(t)\varphi_\alpha(y)=0,\quad y'+q(t)\varphi_\beta(x)=0. \tag{C}$$

For the pioneering systematic investigation of first-order ordinary differential systems including (C) the reader is referred to the book of Mirzov [2].

It should be noticed that the results obtained for system (A) find applications to systems of the form

$$x'(g(t))+p(t)\varphi_\alpha(y(k(t)))=0,\quad y'(h(t))+q(t)\varphi_\beta(x(l(t)))=0,$$

as well as to scalar equations of the form

$$\big(p(t)\varphi_\alpha(x'(g(t)))\big)'+q(t)\varphi_\beta(x(l(t)))=0.$$

References

[1] J. Jaroš and T. Kusano, Asymptotic analysis of positive solutions of first order cyclic functional differential systems. *Georgian Math. J.* (to appear).

[2] J. D. Mirzov, Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations. *Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica*, 14. *Masaryk University, Brno*, 2004.