# Unique Solvability and Additive Averaged Rothe's Type Scheme for One Nonlinear Multi-Dimensional Integro-Differential Parabolic Problem

**Temur Jangveladze**

*I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;*
*Georgian Technical University, Tbilisi, Georgia*
*E-mail:* `tjangv@yahoo.com`

The paper is devoted to the existence and uniqueness of a solution of the initial-boundary problem for one nonlinear multi-dimensional integro-differential equation of parabolic type. Construction and study of the additive averaged Rothe's type scheme is also given. The studied equation is based on well-known Maxwell's system arising in mathematical simulation of electromagnetic field penetration into a substance [10]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of magnetic field, $\theta$ is temperature, $c_\nu$ and $\nu_m$ characterize correspondingly heat capacity and electroconductivity of the medium.

The system $(1), (2)$ is complex and its investigation and numerical resolution still yield for special cases (see, for example, [6] and the references therein).

In [1], the Maxwell's system $(1), (2)$ were proposed to integro-differential form

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int_0^t |\operatorname{rot} H|^2 \, d\tau\right) \operatorname{rot} H\right], \tag{3}$$

where $a = a(S)$ is dependent on coefficients $c_\nu$, $\nu_m$ and is defined for $S \in [0, \infty)$.

Making certain physical assumptions in mathematical description of the above-mentioned process in [12], a new integro-differential model is constructed which represents a generalization of the system (3)

$$\frac{\partial H}{\partial t} = a\left(\int_\Omega \int_0^t |\operatorname{rot} H|^2 \, dx \, d\tau\right) \Delta H. \tag{4}$$

Principal characteristic peculiarity of systems (3) and (4) is connected with the appearance in the coefficient with derivative of higher order nonlinear term depended on the integral of time and space variables. These circumstances requires different discussions than it is usually necessary for the solution of local differential problems.

The literature on the questions of existence, uniqueness, and regularity of solutions to the models of above types is very rich. In [1–5, 11–13], the solvability of the initial-boundary value problems for (3) type models in scalar cases is studied using a modified version of the Galerkin's method and compactness arguments that are used in [14, 16] for investigation elliptic and parabolic equations. The uniqueness of solutions is investigated also in works [1–5, 11–13]. The asymptotic behavior of solutions is discussed in [4, 6, 9] and in a number of other works as well. Note also that to numerical resolution of (3) and (4) type one-dimensional models were devoted many works as well (see, e.g., [5–7, 9] and the references therein).

Many authors study the Rothe's scheme, semi-discrete scheme with space variable, finite element and finite difference approximation for a integro-differential models (see, for example, [5–9, 14, 15]).

It is very important to study decomposition analogs for above-mentioned multi-dimensional differential and integro-differential models as well. At present there are some effective algorithms for solving the multi-dimensional problems (see, for example, [14, 15] and the references therein).

This paper dedicated to the existence and uniqueness of solutions of initial-boundary value problem. Investigations are given in usual Sobolev spaces. Main attention is also paid to investigation of Rothe's type additive averaged scheme. In this paper we shall focus our attention to (4) type multi-dimensional integro-differential scalar equation.

Let $\Omega$ is bounded domain in the $n$-dimensional Euclidean space $R^n$ with sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0, T)$ of the variables $(x, t) = (x_1, x_2, \dots, x_n, t)$ let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \left(1 + \int_{\Omega} \int_{0}^{t} \left|\frac{\partial U}{\partial x_i}\right|^2 dx\, d\tau\right) \frac{\partial^2 U}{\partial x_i^2} = f(x, t), \quad (x, t) \in Q, \tag{5}$$

$$U(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \tag{6}$$

$$U(x, 0) = 0, \quad x \in \overline{\Omega}, \tag{7}$$

where $T$ is a fixed positive constant, $f$ is a given function of its arguments.

Since problem (5)–(7) similar to problems considered in [4], where investigation of (3) type multi-dimensional scalar equations is given and at first is discussed unique solvability and asymptotic behavior of (5) type models as well, we can follow the same procedure used there. Using modified version of the Galerkin's method and compactness arguments [16], [14] the following statement can be proved.

**Theorem 1.** *If*

$$f \in W_2^1(Q), \quad f(x, 0) = 0,$$

*then there exists a unique solution $U$ of problem (5)–(7) satisfying the properties:*

$$U \in L_4(0, T; \overset{\circ}{W}{}_4^1(\Omega)) \cap L_2(0, T; W_2^2(\Omega)), \quad \frac{\partial U}{\partial t} \in L_2(Q),$$

$$\sqrt{T - t}\, \frac{\partial^2 U}{\partial t \partial x_i} \in L_2(Q), \quad i = 1, \dots, n.$$

The proof of the formulated theorem is divided into several steps. One of the basic step is to obtain necessary a priori estimates.

Using the scheme of investigation as in, e.g., [4, 6, 9], it is not difficult to get the result of exponentially asymptotic behavior of solution as $t \to \infty$ for (5) equation with $f(x, t) \equiv 0$ and homogeneous boundary (6) and nonhomogeneous initial (7) conditions.

On $[0, T]$ let us introduce a net with mesh points denoted by $t_j = j\tau$, $j = 0, 1, \dots, J$, with $\tau = 1/J$.

Coming back to problem (5)–(7), let us construct additive averaged Rothe's type scheme:

$$\begin{gathered} \eta_i \, \frac{u_i^{j+1} - u^j}{\tau} = \left(1 + \tau \sum_{k=1}^{j+1} \int_{\Omega} \left|\frac{\partial u_i^k}{\partial x_i}\right|^2 dx\right) \frac{\partial^2 u_i^{j+1}}{\partial x_i^2} + f_i^{j+1}, \\ u_i^0 = u^0 = 0, \quad i = 1, \dots, n, \quad j = 0, 1, \dots, J - 1, \end{gathered} \tag{8}$$

with homogeneous boundary conditions, where $u_i^j(x)$, $j=1,\dots,J$, is a solution of problem (8) and the following notations are introduced:

$$u^j(x)=\sum_{i=1}^{n}\eta_i u_i^j(x),\quad \sum_{i=1}^{n}\eta_i=1,\ \ \eta_i>0,\quad \sum_{i=1}^{n}f_i^{j+1}(x)=f^{j+1}(x)=f(x,t_{j+1}),$$

where $u^j$ denotes approximation of exact solution $U$ of problem (5)–(7) at $t_j$. We use usual norm $\|\cdot\|$ of the space $L_2(\Omega)$.

**Theorem 2.** *If problem* (5)–(7) *has sufficiently smooth solution, then the solution of problem* (8) *converges to the solution of problem* (5)–(7) *and the following estimate is true*

$$\|U^j-u^j\|=O(\tau^{1/2}),\ \ j=1,\dots,J.$$

Using early investigated finite difference and finite element schemes for one-dimensional (5) type models (see, for example, [5–7, 9]) now we can reduce numerical resolution of the multi-dimensional integro-differential model (5) to one-dimensional ones. It is very important to construct and investigate studied in this note type models for more general type nonlinearities and for (5) type multi-dimensional systems as well.

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