Existence of Optimal Control on an Infinitive Interval for Systems of Differential Equations with Pulses at Non-Fixed Times

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We consider two problems of optimal control for systems of differential equations with pulse action

$$\begin{gathered}\dot{x}=A(x,t)+B(x,t)u, \ \ x\notin S,\\ \Delta x\big|_{x\in S}=g(x),\\ x(0)=x_0.\end{gathered}\tag{1}$$

In the first problem for the system (1) the quality criteria is the following

$$J(u)=\int\limits_0^\infty \nu(t)L(t,x(t),u(t))\,dt\to\inf,\tag{2}$$

where S – some hypersurface in the space R^d, $x_0\in R^d$ – a fixed vector, $t\in[0,\infty)$, $x\in R^d$, $L(t,x,u)$ – a limited function, $u\in U\subset R^m$, U – a closed, convex set in the space R^m, $0\in U$, $A(x,t)$ – d-dimensional vector function, $B(x,t)$ – $d\times m$-dimensional matrix, g – d-dimensional vector function.

In the second problem for the system (1) we consider the quality criteria

$$J(u)=\int\limits_0^\theta \nu(t)L(t,x(t),u(t))\,dt\longrightarrow\inf,\tag{3}$$

where $t\in[0,\infty)$, $x\in D$, D – a limited area in the space R^d, $D\cap S$ – is not empty, $x_0\in R^d$ – a fixed vector, θ – a moment of leaving the solution $x(t)$ the area D.

We consider the problem (1), (2) with the following conditions: functions $A(x,t)$, $B(x,t)$ are continuous for a set of variables $t\in[0,\infty)$, $x\in R^d$, $g(x)$ is continuous by $x\in R^d$ and the condition of Lipschitz is satisfied, there is a constant $H>0$ such that for any $x_1,x_2\in R^d$, $t\geq 0$ and $u\in U$ the conditions:

$$\big|A(t,x_1)-A(t.x_2)\big|\leq H|x_1-x_2|, \quad \big\|B(t,x_1)-B(t.x_2)\big\|\leq H|x_1-x_2|\tag{4}$$

hold.

Functions $L(t,x,u)$, $L_x(t,x,u)$ and $L_u(t,x,u)$ are continuous for a set of variables, for any $t\in[0,\infty)$, $x\in R^d$ and $u\in U$, the following conditions are satisfied:

1) $L(t,x,u)\geq 0$ for any $t\in[0,\infty)$, $x\in R^d$ and $u\in U$;

2) there are constants $R > 0$ and $p > 2$ such that for any $t \in [0, \infty)$, $x \in R^d$ and $u \in U$, the inequality
$$L(t, x, u) \geq R(1 + |u|^p)$$
is fulfilled;

3) there is $M > 0$ such that for any $t \in [0, \infty)$, $x \in R^d$ and $u \in U$,
$$|L_x(t, x, u)| + |L_u(t, x, u)| \leq M(1 + |u|^{p-1});$$

4) $L(t, x, u)$ is convex by u for any fixed $t \in [0, \infty)$, $x \in R^d$.

For the problem $(1), (3)$ conditions are similar to the problem $(1), (2)$ for $x \in D$.
Acceptable for problems $(1), (2)$ and $(1), (3)$ are such controls $u = u(t)$ that:

(a) $u(t) \in L_p([0, \infty))$, $u(t) \in U$, $t \in [0, \infty)$;

(b) there is a constant $C_1 > 0$ which does not depend on $u(t)$ and the following condition holds:
$$\int\limits_0^\infty |u(t)|^p \, dt \leq C_1.$$

The set of acceptable controls will be named acceptable for (1), (2) and (1), (3) and will be denoted by F.

We assume that the hypersurface S is a compact set and is given by $s(x) = 0$, where s is a continuous function.

Let τ_u^k be moments in which the solution $x(t, u)$ hit on the hypersurface S.

Theorem 1. *Let the system* (1) *with the quality criteria* (2)*, for functions $A(x, t)$, $B(x, t)$, $\nu(t)$ and $L(t, x, u)$ satisfy the condition* (4) *and* 1)–3)*, the function $\nu(t) \in L_1([0, \infty))$, $0 \leq \nu(t) \leq 1$ for any $t \geq 0$. Then the problem* $(1), (2)$ *has a solution in the set of acceptable controls F.*

Theorem 2. *Let the system* (1) *with the quality criteria* (3)*, for functions $A(x, t)$, $B(x, t)$, $\nu(t)$ and $L(t, x, u)$ satisfy the condition of Theorem* 1 *for $t \geq 0$, $x \in D$. Then the problem* $(1), (3)$ *has a solution in the set of acceptable controls F.*

Proof for the problem $(1), (2)$. Since $J(u) \geq 0$, then there exists a non-negative lower bound m of values $J(u)$. Let u_n be the sequence of acceptable controls such that: $J(u_n) \to m$, $n \to \infty$. Namely,
$$J(u_n) = \int\limits_0^\infty \nu(t) L(t, x_n(t), u_n(t)) \, dt \longrightarrow m, \quad n \to \infty,$$
where $x_n(t)$ are solutions of the system (1) which correspond to controls $u_n(t)$.

The condition (b) guarantees a weak compactness of the sequence $u_n(t)$. Thus the sequence $u_n(t)$ converge weakly to $u^*(t) \in L_p([0, \infty))$. It is easy to show that $u^*(t) \in U$ for almost all $t \in [0, \infty)$.

We take an arbitrary $T > 0$ and fix. Since in the interval $[0, T]$ all the conditions of the Theorem 1 are fulfilled, then there exists $x_T^*(t)$ – the solution of the system (1) at $[0, T]$, which correspond to control $u^*(t)$ and $x_n(t) \rightrightarrows x_T^*(t)$, $n \to \infty$ for any $t \in [0, T]$.

We show that there is a subsequence of functions $x_{n_n}(t)$ which pointwise converges to the function $x^*(t)$ for any $t \in [0, \infty)$.

For $T=1$ there exists the subsequence $x_{n_1}(t)$ of the sequence $x_{n_n}(t)$, $n\geq 1$ such that $x_{n_1}(t)\rightrightarrows x_1^*(t)$ for any $t\in[0,1]$.

For $T=2$ there exists the subsequence $x_{n_2}(t)$ of the sequence $x_{n_1}(t)$, $n\geq 1$ such that $x_{n_2}(t)\rightrightarrows x_2^*(t)$ for any $t\in[0,2]$, where $x_2^*(t)=x_1^*(t)$, $t\in[0,1]$.

Similarly, for any natural N there exists the subsequence $x_{n_N}(t)$ of the sequence $x_{n_{N-1}}(t)$ such that $x_{n_N}(t)\rightrightarrows x_N^*(t)$ for any $t\in[0,N]$, where $x_N^*(t)=x_{N-1}^*(t)$, $t\in[0,N-1]$.

Using the diagonal method of this sequences, we can distinguish the following subsequence $x_{n_n}(t)$, $n\geq 1$

$$x_{1_1}(t),x_{2_2}(t),x_{3_3}(t),\dots,x_{n_n}(t),\dots.$$

This sequence pointwise converges to the function $x^*(t)$ for any $t\in[0,\infty)$.

Similarly to [3], it can be shown that the control $u^*(t)$ is optimal for the problem (1), (2), that $J(u^*)=m$.

Proof for the problem (1), (3). The proof of Theorem 2 is similar to the proof of Theorem 1, but it must be taken into account the moment of coming out the solution of the area.

References

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