Invariant Tori and Dichotomy of Linear Extension of Dynamical Systems

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1 Introduction and preliminaries

We consider a system of differential equations defined in the direct product of a torus $\mathcal{T}_m$, $m \in \mathbb{N}$ and an Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$,

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{1.1}$$

where $(\varphi_1, \dots, \varphi_m)^\intercal \in \mathcal{T}_m$, $(x_1, \dots, x_n)^\intercal \in \mathbb{R}^n$, $a \in \mathcal{C}^1(\mathcal{T}_m)$ is an m-dimensional vector function, $A, f \in \mathcal{C}(\mathcal{T}_m)$ are $n \times n$ square matrix and n-dimensional vector function respectively; $\mathcal{C}^r(\mathcal{T}_m)$ stands for the space of continuously differentiable up to the order r 2π-periodic with respect to each of the variables φ_j, $j = 1, \dots, m$ functions defined on the surface of the torus $\mathcal{T}_m$. The problem of the existence and construction of invariant toroidal manifold

$$x = u(\varphi) \in \mathcal{C}(\mathcal{T}_m), \ \ \varphi \in \mathcal{T}_m$$

of the system (1.1) for any inhomogeneity $f(\varphi) \in \mathcal{C}(\mathcal{T}_m)$ can be solved using a notion of Green–Samoilenko function [7]. The existence of such a function is sufficient for the existence of non-trivial invariant torus for system (1.1). In particular, Green–Samoilenko function exists if for any $\varphi \in \mathcal{T}_m$ the system

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x \tag{1.2}$$

is exponential dichotomous on the entire real axis $\mathbb{R} = (\infty, +\infty)$. This means that there exist a projection matrix $C(\varphi) = C^2(\varphi)$ and constants $K \geq 1$, $\alpha > 0$ that do not depend on φ, τ such that the following inequalities

$$\begin{aligned} \big\|\Omega_0^t(\varphi)C(\varphi)\Omega_\tau^0(\varphi)\big\| &\leq Ke^{-\alpha(t-\tau)}, \ \ t \geq \tau, \\ \big\|\Omega_0^t(\varphi)(I - C(\varphi))\Omega_\tau^0(\varphi)\big\| &\leq Ke^{-\alpha(\tau-t)}, \ \ \tau \geq t \end{aligned} \tag{1.3}$$

are satisfied for any $t, \tau \in \mathbb{R}$. Here $\Omega_\tau^t(\varphi)$ is $(n \times n)$-dimensional fundamental matrix of the system (1.2) such that $\Omega_\tau^\tau(\varphi) \equiv I_n$; $\varphi_t(\varphi)$ is a solution of the initial value problem $\frac{d\varphi}{dt} = a(\varphi)$, $\varphi_0(\varphi) = \varphi$.

In recent papers [3, 5, 6] some particular classes of system (1.1) were distinguished for which the corresponding homogenous equations possess Green–Samoilenko function. These are the systems whose matrix $A(\varphi)$ becomes Hurwitz matrix for φ-s from the non-wandering set of dynamical system $\frac{d\varphi}{dt} = a(\varphi)$. We recall here the definition of non-wandering set.

Definition 1.1. A point φ is called wandering if there exist its neighbourhood $U(\varphi)$ and a positive number $T > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = 0 \ \text{ for } \ t \geq T.$$

Let W be a set of all wandering points of dynamical system and $\Omega = \mathcal{T}_m \setminus W$ be a set of non-wandering points. From the compactness of a torus it follows that the set Ω is nonempty and compact.

Analogously to [5,6], in this paper we also consider the case when matrix $A(\varphi)$ is a constant matrix in non-wandering set Ω: $A(\varphi)|_{\varphi\in\Omega} = \widetilde{A}$. However we do not require the real parts of all eigenvalues of matrix $\widetilde{A}$ to be negative in order to guarantee the existence of invariant toroidal manifold for system (1.1).

2 Main results

To state the main result of the paper we recall that system (1.2) possesses exponential dichotomy property on semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ if there exist projection matrices $C_+(\varphi) = C_+^2(\varphi)$ and $C_-(\varphi) = C_-^2(\varphi)$ and constants $K_1, K_2 \geq 1$, $\alpha_1, \alpha_2 > 0$ that do not depend on φ, τ such that for any $\varphi \in \mathcal{T}_m$ the following inequalities

$$\begin{gathered}
\big\|\Omega_0^t(\varphi)C_+(\varphi)\Omega_\tau^0(\varphi)\big\| \leq K_1 e^{-\alpha_1(t-\tau)}, \ \ t \geq \tau, \\
\big\|\Omega_0^t(\varphi)(I - C_+(\varphi))\Omega_\tau^0(\varphi)\big\| \leq K_1 e^{-\alpha_1(\tau-t)}, \ \ \tau \geq t, \ \ \forall\, t,\tau \in \mathbb{R}_+, \\
\big\|\Omega_0^t(\varphi)C_-(\varphi)\Omega_\tau^0(\varphi)\big\| \leq K_2 e^{-\alpha_2(t-\tau)}, \ \ t \geq \tau, \\
\big\|\Omega_0^t(\varphi)(I - C_-(\varphi))\Omega_\tau^0(\varphi)\big\| \leq K_2 e^{-\alpha_2(\tau-t)}, \ \ \tau \geq t, \ \ \forall\, t,\tau \in \mathbb{R}_-
\end{gathered} \tag{2.1}$$

are satisfied.

Theorem 2.1. *Let matrix $A(\varphi)$ from* (1.1) *be constant in non-wandering set Ω:*

$$A(\varphi)|_{\varphi\in\Omega} = \widetilde{A},$$

and the corresponding linear system $\frac{dx}{dt} = \widetilde{A}x$ be exponential dichotomous on $\mathbb{R}$. Then for any $\varphi \in \mathcal{T}_m$ the corresponding homogenous system $\frac{dx}{dt} = A(\varphi_t(\varphi))x$ is exponential dichotomous on semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$, e.g. there exist projection matrices $C_+(\varphi)$ and $C_-(\varphi)$ such that the inequalities (2.1) *are satisfied and*

$$C_\pm(\varphi_t(\varphi)) = \Omega_0^t(\varphi)C_\pm(\varphi)\Omega_t^0(\varphi), \quad C_\pm^2(\varphi) = C_\pm(\varphi).$$

For example, the conditions of Theorem 2.1 are satisfied in the case when the real parts of all eigenvalues of constant matrix $\widetilde{A}$ are nonzero.

Denote by $D(\varphi) = C_+(\varphi) - (I - C_-(\varphi))$ an $(n \times n)$-dimensional matrix. Let $D^+(\varphi)$ be its Moore–Penrose pseudoinverse [2], and $P_{N(D)}(\varphi)$ and $P_{N(D^*)}(\varphi)$ be $(n \times n)$-orthoprojector matrices

$$\begin{gathered}
P_{N(D)}^2(\varphi) = P_{N(D)}(\varphi) = P_{N(D)}^*(\varphi), \\
P_{N(D^*)}^2(\varphi) = P_{N(D^*)}(\varphi) = P_{N(D^*)}^*(\varphi)
\end{gathered}$$

that project $\mathbb{R}^n$ onto the kernel $N(D) = \ker D(\varphi)$ and co-kernel $N(D^*) = \ker D^*(\varphi)$ of the matrix $D(\varphi)$:

$$P_{N(D^*)}(\varphi) = I - D(\varphi)D^+(\varphi), \quad P_{N(D)}(\varphi) = I - D^+(\varphi)D(\varphi).$$

Theorem 2.1 states that exponential dichotomy on $\mathbb{R}$ property of a "limit system" $\frac{dx}{dt} = \widetilde{A}x$ implies the exponential dichotomy on semiaxes $\mathbb{R}_+$, $\mathbb{R}_-$ for the system $\frac{dx}{dt} = A(\varphi_t(\varphi))x$. Combination of this result with [1,4] immediately leads to the following corollaries.

Corollary 2.1. *Let matrix $A(\varphi)$ from* (1.1) *be constant in non-wandering set Ω:*

$$A(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A},$$

and the corresponding linear system $\frac{dx}{dt} = \widetilde{A}x$ be exponential dichotomous on $\mathbb{R}$. Then system (1.1) *has an invariant toroidal manifold if and only if the inhomogeneity $f(\varphi) \in \mathcal{C}(\mathcal{T}_m)$ satisfies the following constraint*

$$P_{N(D^*)}(\varphi) \int\limits_{-\infty}^{+\infty} C_-(\varphi)\Omega^0_\tau(\varphi) f(\varphi_\tau(\varphi))\, d\tau = 0.$$

Corollary 2.2. *Let matrix $A(\varphi)$ from* (1.1) *be constant in non-wandering set Ω:*

$$A(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A},$$

and the corresponding linear system $\frac{dx}{dt} = \widetilde{A}x$ be exponential dichotomous on $\mathbb{R}$. If additionally for any $\varphi \in \mathcal{T}_m$ matrices $\widetilde{A}$ and $(A(\varphi) - \widetilde{A})$ commute then system (1.1) *has an invariant toroidal manifold for any inhomogeneity $f(\varphi) \in \mathcal{C}(\mathcal{T}_m)$.*

3 Conclusions and discussion

New results that are presented in this paper allow to investigate qualitative behavior of solutions of a class of nonlinear systems that have a simple structure of limit sets and recurrent trajectories. Additionally they can be used to prove the persistence of a stable invariant toroidal manifold under the perturbation of the right-hand side of (1.1) in the case when this perturbation is sufficiently small only in non-wandering set Ω, but not on the whole surface of the torus $\mathcal{T}_m$.

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