# On Estimates for the First Eigenvalue of Some Sturm–Liouville Problems with Dirichlet Boundary Conditions and a Weighted Integral Condition

**S. Ezhak and M. Telnova**

*Plekhanov Russian University of Economics, Moscow, Russia*
*E-mails:* svetlana.ezhak@gmail.com; mytelnova@yandex.ru

## 1 Introduction

In this paper, a problem is considered whose origin was the Lagrange problem. It is a problem on finding the form of the firmest column of given volume. The Lagrange problem was the source for different extremal eigenvalue problems. One of them is the eigenvalue problem for second-order differential equations with an integral condition on the potential.

The Dirichlet problem for the equation $y'' + \lambda Q(x)y = 0$ with non-negative summable on $[0,1]$ function $Q(x)$ satisfying $\int\limits_0^1 Q^\gamma(x)\,dx = 1$, as $\gamma \in \mathbb{R}$, $\gamma \neq 0$, was considered in [1]. The Dirichlet problem for the equation $y'' - Q(x)y + \lambda y = 0$ with a real integrable on $(0,1)$ by Lebesgue function $Q$ was considered in [8] for $\gamma \geqslant 1$.

In this paper, the problems of that kind are considered under different integral conditions, in particular, if the integral condition contains a weight function. The purpose of research is to give methods of finding the sharp estimates for the first eigenvalue of Sturm–Liouville problems with Dirichlet boundary conditions for those values of the integral condition parameters for which the estimates are finite, and to prove attainability of those estimates.

Consider the Sturm–Liouville problem

$$y'' + \sigma Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where $\sigma = \pm 1$, and $Q$ belongs to the set $T_{\alpha,\beta,\gamma}$ of all real–valued locally integrable functions on $(0,1)$ with non–negative values such that the following integral condition holds

$$\int\limits_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)\,dx = 1, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \gamma \neq 0. \tag{3}$$

A function $y$ is a *solution* to problem $(1),(2)$ if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative $y'$ is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

A function $y \in H_0^1(0,1)$ is called a *weak solution* to equation (1) if for any function $\psi \in C_0^\infty(0,1)$ the following equality

$$\int\limits_0^1 (y'\psi' + \sigma\, Q(x)y\psi)\,dx = \lambda \int\limits_0^1 y\psi\,dx$$

holds.

We give estimates for

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q),\quad M_{\alpha,\beta,\gamma}=\sup_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q).$$

For any function $Q\in T_{\alpha,\beta,\gamma}$ by $H_Q$ we denote the closure of the set $C_0^\infty(0,1)$ in the norm

$$\|y\|_{H_Q}=\bigg(\int_0^1 y'^2\,dx+\int_0^1 Q(x)y^2\,dx\bigg)^{\frac12}.$$

For any function $Q\in T_{\alpha,\beta,\gamma}$ it is proved (see, for example, [5,6]) that

$$\lambda_1(Q)=\inf_{y\in H_Q\setminus\{0\}}R[Q,y],\ \text{ where }\ R[Q,y]=\frac{\int_0^1(y'^2-\sigma Q(x)y^2)\,dx}{\int_0^1 y^2\,dx}\,.$$

Previous results are published in [2–7]. Results of this type can be useful to give methods of finding the sharp estimates for eigenvalues in cases of non-differentiable functionals.

## 2 Main results

### 2.1 Estimates for $\sigma=-1$

By Friedrichs' inequality for any function $Q\in T_{\alpha,\beta,\gamma}$ we obtain

$$\inf_{y\in H_Q\setminus\{0\}}\frac{\int_0^1 y'^2\,dx+\int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}\geqslant\inf_{y\in H_Q\setminus\{0\}}\frac{\int_0^1 y'^2\,dx}{\int_0^1 y^2\,dx}\geqslant\inf_{y\in H_0^1(0,1)\setminus\{0\}}\frac{\int_0^1 y'^2\,dx}{\int_0^1 y^2\,dx}=\pi^2.$$

Consequently, for any $\alpha,\beta,\gamma\in\mathbb{R}$, $\gamma\neq0$, we have

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\inf_{y\in H_Q\setminus\{0\}}R[Q,y]\geqslant\pi^2.$$

If $\gamma>0$, then it is proved that $m_{\alpha,\beta,\gamma}=\pi^2$ (see, for example, [5,6]).

Put $\gamma<0$. For any positive function $Q\in T_{\alpha,\beta,\gamma}$ by the Hölder inequality we have

$$\int_0^1 Q(x)y^2\,dx\geqslant\bigg(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}]\,dx\bigg)^{\frac{\gamma-1}{\gamma}}. \tag{4}$$

Consider the subspace $B_{\alpha,\beta,\gamma}$ of functions in the space $H_0^1(0,1)$ such that

$$\|y\|^2_{B_{\alpha,\beta,\gamma}}=\int_0^1 y'^2\,dx+\bigg(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\bigg)^{\frac{\gamma-1}{\gamma}}<+\infty.$$

By inequality (4) we have $H_Q \subset B_{\alpha,\beta,\gamma} \subset H_0^1(0,1)$. Put $m = \inf\limits_{y \in B_{\alpha,\beta,\gamma} \setminus \{0\}} G[y]$, where

$$G[y] = \frac{\int\limits_0^1 y'^2\,dx + \Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y^2\,dx}\,.$$

Since

$$\inf_{y \in H_Q \setminus \{0\}} R[Q,y] \geqslant \inf_{y \in H_Q \setminus \{0\}} G[y] \geqslant \inf_{y \in B_{\alpha,\beta,\gamma} \setminus \{0\}} G[y] = m,$$

it follows that

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \geqslant \inf_{y \in H_Q \setminus \{0\}} G[y] \geqslant \inf_{y \in B_{\alpha,\beta,\gamma} \setminus \{0\}} G[y] = m.$$

The following two theorems prove that $m_{\alpha,\beta,\gamma} = m$.

Consider the set

$$\Gamma = \left\{ y \in B_{\alpha,\beta,\gamma} \;\Big|\; \int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx = 1 \right\}.$$

**Theorem 2.1.** *If $\gamma < 0$, then there exists a non-negative function $u \in \Gamma$ such that $G[u] = m$, moreover, for $\gamma < -1$ $u$ is a weak solution to the equation*

$$u'' + mu = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}}.$$

**Theorem 2.2.** *Suppose that $\gamma < 0$ and the function $u$ satisfies the conditions of Theorem 2.1. Then there exists a sequence $Q_n(x) \in T_{\alpha,\beta,\gamma}$ such that $R[Q_n,u] \to G[u] = m$ as $n \to \infty$ and $m_{\alpha,\beta,\gamma} = m$.*

**Remark 2.1.** In the case of $\gamma < 0$, inequalities for $m_{\alpha,\beta,\gamma} = m$ can be found, for example, in [5,6].

**Theorem 2.3** (see [2,6,7])**.** *For $M_{\alpha,\beta,\gamma}$ the following estimates hold:*

1. *If $\gamma < 0$ or $0 < \gamma < 1$, then we have $M_{\alpha,\beta,\gamma} = \infty$.*
2. *If $\gamma \geqslant 1$, then we have $M_{\alpha,\beta,\gamma} < \infty$, moreover:*

   1) *If $\gamma > 1$, then there is a function $Q_* \in T_{\alpha,\beta,\gamma}$ and a positive on $(0,1)$ function $u \in H_{Q_*}$ such that $R[Q_*,u] = G[u] = m$ and $M_{\alpha,\beta,\gamma} = m > \pi^2$. The function $u$ satisfies the equation*

   $$u'' + mu = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}}$$

   *and the condition*

   $$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx = 1.$$

   *In the case of $\gamma > 1$, $\alpha = \beta = 0$, $m$ is the solution of the system of the equations*

   $$\begin{cases} \displaystyle\int\limits_0^H \frac{du}{\sqrt{mH^2 - mu^2 - \frac{2}{p}H^p + \frac{2}{p}u^p}} = \frac{1}{2}\,, \\ \displaystyle\int\limits_0^H \frac{u^p}{\sqrt{mH^2 - mu^2 - \frac{2}{p}H^p + \frac{2}{p}u^p}}\,du = \frac{1}{2}\,, \end{cases}$$

*where* $H=\max\limits_{x\in[0,1]} u(x)$, $p=\frac{2\gamma}{\gamma-1}\,s$.

2) *If* $\gamma\geqslant 1$ *and* $\alpha,\beta>\gamma$, *then we have* $M_{\alpha,\beta,\gamma}\leqslant R[\frac{1}{y_1^2},y_1]$, *where* $y_1(x)=x^{\frac{\alpha}{2\gamma}}(1-x)^{\frac{\beta}{2\gamma}}$.

3) *If* $\beta\leqslant\gamma<\alpha$ *and* $y_2(x)=x^{\frac{\alpha}{2\gamma}}\sin\pi(1-x)$, *then we have*

$$M_{\alpha,\beta,\gamma}\leqslant\frac{\int\limits_0^1 {y_2'}^2\,dx+\pi^2\big(\frac{\gamma-1}{3\gamma-\beta-1}\big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y_2^2\,dx}\quad\text{for }\ \gamma>1,$$

$$M_{\alpha,\beta,\gamma}\leqslant\frac{\int\limits_0^1 {y_2'}^2\,dx+\pi^2}{\int\limits_0^1 y_2^2\,dx}\quad\text{for }\ \gamma=1.$$

*If* $\alpha\leqslant\gamma<\beta$ *and* $y_3(x)=(1-x)^{\frac{\beta}{2\gamma}}\sin\pi x$, *then we have*

$$M_{\alpha,\beta,\gamma}\leqslant\frac{\int\limits_0^1 {y_3'}^2\,dx+\pi^2\big(\frac{\gamma-1}{3\gamma-\beta-1}\big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y_3^2\,dx}\quad\text{for }\ \gamma>1,$$

$$M_{\alpha,\beta,\gamma}\leqslant\frac{\int\limits_0^1 {y_3'}^2\,dx+\pi^2}{\int\limits_0^1 y_3^2\,dx}\quad\text{for }\ \gamma=1.$$

4) *If* $\gamma\geqslant 1$, *then*

   (a) *for* $\alpha>\gamma$, $\beta\leqslant 0$ *and* $y_2(x)=x^{\frac{\alpha}{2\gamma}}\sin\pi(1-x)$ *we have* $M_{\alpha,\beta,\gamma}\leqslant R[\frac{1}{y_2^2},y_2]$;

   (b) *for* $\beta>\gamma$, $\alpha\leqslant 0$ *and* $y_3(x)=(1-x)^{\frac{\beta}{2\gamma}}\sin\pi x$ *we have* $M_{\alpha,\beta,\gamma}\leqslant R[\frac{1}{y_3^2},y_3]$.

5) *If* $\gamma=1\geqslant\alpha>0\geqslant\beta$ *or* $\gamma=1\geqslant\beta>0\geqslant\alpha$, *then* $M_{\alpha,\beta,\gamma}\leqslant 2\pi^2$.

6) *If* $\gamma=1\geqslant\alpha$, $\beta>0$, *then* $M_{\alpha,\beta,\gamma}\leqslant 3\pi^2$.

7) *If* $\gamma=1$, $\alpha,\beta\leqslant 0$, *then* $M_{\alpha,\beta,\gamma}\leqslant\frac{5}{4}\,\pi^2$. *If* $\gamma=1$, $\alpha=\beta=0$, *then there exist functions* $Q_*(x)\in T_{0,0,1}$ *and* $u\in H_0^1(0,1)$ *such that*

$$M_{0,0,1}=R[Q_*,u]=\frac{\pi^2}{2}+1+\frac{\pi}{2}\,\sqrt{\pi^2+4}\,.$$

**Remark 2.2.** In the case of $\gamma>1$, inequalities for $M_{\alpha,\beta,\gamma}=m$ can be found, for example, in [6,7]. In the case of $\gamma=1$, attainability of sharp estimates for $M_{\alpha,\beta,1}$ were proved in [10].

## 2.2 Estimates for $\sigma=1$

**Theorem 2.4.** 1. *For any* $\alpha,\beta,\gamma\in\mathbb{R}$, $\gamma\neq 0$, *we have* $M_{\alpha,\beta,\gamma}\leqslant\pi^2$.

2. *If* $\gamma > 1$*, then* $M_{0,0,\gamma} = \pi^2$ *and there exist functions* $Q_*(x) \in T_{0,0,\gamma}$ *and* $u \in H_0^1(0,1)$ *such that* $m_{0,0,\gamma} = R[Q_*, u] \geqslant \frac{\pi^2}{2}$ .

3. *If* $\gamma = 1$*, then* $M_{0,0,1} = \pi^2$*,* $m_{0,0,1} = \lambda_*$*, where* $\lambda_* \in (0, \pi^2)$ *is the solution to the equation* $2\sqrt{\lambda} = \mathrm{tg}(\frac{\sqrt{\lambda}}{2})$*. Here* $m_{0,0,1}$ *is attained at* $Q(x) = \delta(x - \frac{1}{2})$*.*

4. *If* $\frac{1}{2} \leqslant \gamma < 1$*, then for any* $\alpha, \beta, \gamma \in \mathbb{R}$*,* $\gamma \neq 0$*, we have* $m_{\alpha,\beta,\gamma} = -\infty$*,* $M_{0,0,\gamma} = \pi^2$*.*

5. *If* $\frac{1}{3} \leqslant \gamma < 1/2$*, then for any* $\alpha, \beta, \gamma \in \mathbb{R}$*,* $\gamma \neq 0$*, we have* $m_{\alpha,\beta,\gamma} = -\infty$*,* $M_{0,0,\gamma} \leqslant \pi^2$*.*

6. *If* $0 < \gamma < \frac{1}{3}$ *, then for any* $\alpha, \beta, \gamma \in \mathbb{R}$*,* $\gamma \neq 0$*, we have* $m_{\alpha,\beta,\gamma} = -\infty$*,* $M_{0,0,\gamma} < \pi^2$*.*

7. *If* $\gamma < 0$*, then for any* $\alpha, \beta, \gamma \in \mathbb{R}$*,* $\gamma \neq 0$*, we have* $m_{\alpha,\beta,\gamma} = -\infty$*,* $M_{0,0,\gamma} < \pi^2$*, and there exist functions* $Q_*(x) \in T_{0,0,\gamma}$ *and* $u \in H_0^1(0,1)$ *such that* $M_{0,0,\gamma} = R[Q_*, u]$*.*

**Remark 2.3.** The result $M_{0,0,\gamma} < \pi^2$ for $0 < \gamma < 1/2$ was obtained in [9].

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