Oscillation and Nonoscillation Results for Half-Linear Equations with Deviated Argument

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This is an enlarged abstract of the joint work with Alois Kufner and Komil Kuliev [**?**]. We introduce oscillatory and nonoscillatory criteria for half-linear equations with deviated argument and dedicate it to the 100 birthday anniversary of Professor A. Bitsadze. Our method relies on the *weighted Hardy inequality.*

Let us consider the *half-linear equation* with *deviated argument*

$$\big(r(t)|u'(t)|^{p-2}u'(t)\big)' + c(t)|u(\tau(t))|^{p-2}u(\tau(t)) = 0, \ \ t \in (0,\infty), \tag{1}$$

where $p > 1$, $c : [0,\infty) \to (0,\infty)$ is continuous, $c \in L^1(0,\infty)$, $r : [0,\infty) \to (0,\infty)$ is continuously differentiable, $\tau : [0,\infty) \to \mathbb{R}$ is continuously differentiable and increasing function satisfying $\lim\limits_{t\to\infty} \tau(t) = \infty$.

Assume that (1) has at least one nonzero *global solution* defined on the entire interval $(0,\infty)$. We say that a global solution of (1) is *nonoscillatory* (at ∞) if there exists $T > 0$ such that $u(t) \neq 0$ for all $t > T$. Otherwise, it is called *oscillatory*, i.e., there exists a sequence $\{t_n\}_{n=1}^\infty$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and $u(t_n) = 0$ for all $n \in \mathbb{N}$. We let $p' = \frac{p}{p-1}$.

Theorem 1 (nonoscillatory criterion)**.** *Let*

$$\limsup_{t\to\infty} \bigg(\int\limits_0^t r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} < \frac{(p-1)}{p^{p'}} \tag{2}$$

and

$$\limsup_{t\to\infty} \bigg(\int\limits_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} < \frac{(p-1)}{p^{p'}}\,. \tag{3}$$

Then every global solution of (1) *is nonoscillatory.*

Theorem 2 (oscillatory criterion)**.** *Let one of the following three cases occur:*

(i) *There exists $T > 0$ such that for all $t \geq T$ we have $\tau(t) \geq t$ and*

$$\limsup_{t\to\infty} \bigg[\bigg(\int\limits_0^t r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} + \bigg(\int\limits_t^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_{\tau(t)}^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}}\bigg] > 1.$$

(ii) *There exists $T > 0$ such that for all $t \geq T$ we have $\tau(t) \leq t$ and*

$$\limsup_{t\to\infty} \bigg(\int\limits_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^\infty c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} > 1.$$

(iii) *For any $T > 0$ the function $\tau(t) - t$ changes sign in (T, ∞) and either*

$$\liminf_{\substack{t\to\infty\\ t>\tau(t)}} \bigg(\int\limits_0^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} > 1$$

or

$$\liminf_{\substack{t\to\infty\\ t<\tau(t)}} \Bigg[\bigg(\int\limits_0^{t} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} + \bigg(\int\limits_t^{\tau(t)} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_{\tau(t)}^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}}\Bigg] > 1.$$

Then every global solution of (1) *is oscillatory.*

A typical example of $\tau = \tau(t)$ is a linear function

$$\tau(t) = t - \tau, \ \ \tau \ge 0 \ \text{ is fixed.}$$

Then (1) is half-linear equation with the *delay* given by fixed parameter $\tau \ge 0$. For this, rather special case, (2) implies (3), and only the case (ii) of Theorem 2 occurs. Hence we have the following corollary concerning the equation

$$\big(r(t)|u'(t)|^{p-2}u'(t)\big)' + c(t)|u(t-\tau)|^{p-2}u(t-\tau) = 0, \ \ t \in (0, \infty). \tag{4}$$

Corollary 3 (equation with delay)**.** *Let* (2) *hold. Then every global solution of* (4) *with the delay $\tau \ge 0$ is nonoscillatory. On the other hand, let*

$$\limsup_{t\to\infty} \bigg(\int\limits_0^{t-\tau} r^{1-p'}(s)\,\mathrm{d}s\bigg)\bigg(\int\limits_t^{\infty} c(s)\,\mathrm{d}s\bigg)^{\frac{1}{p-1}} > 1.$$

Then every global solution of (4) *with the delay $\tau \ge 0$ is oscillatory.*

Remark 4. Let us note that nonoscillatory criteria are rare in the literature even for the linear equations with the delay. Oscillatory criteria for solutions of half-linear equations with the delay are presented in recent papers [3]–[**?**], [8] and [**?**]. The methodology in these articles is based on the so-called Riccati technique and the assumptions are different than those of ours. In particular, if $\tau(t) = t$ in (1), we have the "classical" half-linear equation considered e.g. in [1, Chapter 3]. Then oscillatory criterion in Corollary 3 (with $\tau = 0$) recovers [1, Theorem 3.1.2]. On the other hand, nonoscillatory criterion in Corollary 3 (with $\tau = 0$) recovers [1, Theorem 3.1.3]. The approach in [1, Chapter 1] is based also on the *Riccati technique*. In contrast with works on half-linear equations with the delay mentioned above, we present both oscillatory and nonoscillatory criteria and our method relies on the *weighted Hardy inequality*. Similar approach to that of ours was used in [9] to prove oscillation and nonoscillation results for solutions of higher order half-linear equations, but without the deviated argument. For the completeness, we refer also to the papers [**?**], [**?**] and [12] which deal with the half-linear equations with the deviated argument in the case $r(t) = 1$.

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