The Dirichlet Problem for a Class of Anisotropic Mean Curvature Equations

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1 Introduction

We are concerned with the study of the existence, uniqueness, regularity and boundary behaviour of the solutions of the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a > 0$, $b > 0$ are given constants and Ω is a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial\Omega$.

Problem (1.1) has been recently introduced in order to describe the geometry of the human cornea. We refer to [13–17] for the derivation of the model, further discussions on the subject and an additional bibliography. It should however be pointed out that in [13,14,16,17] a simplified version of (1.1) has been investigated, where the curvature operator, $\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2})$, is replaced by its linearization around 0, $\operatorname{div}(\nabla u) = \Delta u$, and, furthermore, Ω is supposed to be either an interval in $\mathbb{R}$, or a disk in $\mathbb{R}^2$. In [2,3] we have instead considered the complete model (1.1) and we have proved the existence of a unique classical solution for any choice of the positive parameters a, b, but still assuming that Ω is an interval in $\mathbb{R}$, or a ball in $\mathbb{R}^N$. Some numerical experiments for approximating the solution of the 1-dimensional problem have also been performed in [2,15]. Later on, in [4], we tackled the quite challenging problem in arbitrary Lipschitz domains and we proved, for all $a, b > 0$, the existence and the uniqueness of a generalized solution, which is regular in the interior and attains the Dirichlet boundary data under an additional condition that relates the values of the parameters with the geometry of the domain. The necessity of considering generalized solutions in this context is dictated by the possible occurrence of solutions which are singular at the boundary, namely solutions that are regular in the interior, but do not attain the Dirichlet condition at some points of the boundary, where in addition the normal derivative blows up. We refer to the survey paper [5] for a thorough discussion of this matter. The following notions of solution for problem (1.1), partially inspired by [6, 7, 9–12, 19], are therefore introduced.

Definition 1.1. A function $u \in W^{1,1}(\Omega)$ is a *generalized* solution of (1.1) if the following conditions hold:

- $\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^N(\Omega)$;
- u satisfies the equation in (1.1) a.e. in Ω;
- for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$,
  - either $u(x) = 0$,
  - or $u(x) > 0$ and $\Big[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,, \nu\Big](x) = -1$,
  - or $u(x) < 0$ and $\Big[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,, \nu\Big](x) = 1$,

  where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and $[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \nu] \in L^\infty(\partial\Omega)$ is the weakly defined trace on $\partial\Omega$ of the component of $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ with respect to the unit outer normal ν to Ω (cf. [1]).

A generalized solution u of (1.1) is *classical* if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u(x) = 0$ on $\partial\Omega$.
A generalized solution u of (1.1) is *singular* if it is not classical.

The concept of generalized solution expressed by Definition 1.1 looks rather natural in the frame of (1.1) and can heuristically be interpreted as follows: the solution u is not required to satisfy the homogeneous Dirichlet boundary condition at all points of $\partial\Omega$, but at any point of $\partial\Omega$ where the zero boundary value is not attained the unit upper normal $\mathcal{N}(u)$ to the graph of u equals either the unit outer normal $(\nu, 0)$, or the unit inner normal $(-\nu, 0)$, according to the sign of u; in this case, roughly speaking, the graph of the solution might be smoothly continued by vertical segments up to the zero level. This kind of boundary behaviour of solutions of the N-dimensional prescribed mean curvature equation has already been observed and discussed in [6, 7, 10, 12]. With reference to Definition 1.1 we can state various existence, uniqueness and regularity results, which are the contents of the next sections.

2 Radially symmetric solutions

Since the equation in (1.1) is invariant under orthogonal transformations, it is natural to look for radially symmetric solution whenever the domain is either a ball, or a spherical shell. However the solvability patterns in the two cases are quite different.

Classical solutions on balls

Let $B = B(x_0, R)$ be the open ball in $\mathbb{R}^N$ of center x_0 and radius R.

Theorem 2.1. *For every $a > 0$, $b > 0$, there exists a unique generalized solution u of* (1.1)*, with $\Omega = B$, which is radially symmetric and classical, with $u \in C^2(\overline{B})$. Moreover, there exists a function $v \in C^2([0,R])$, with $u(x) = v(|x - x_0|)$ for all $x \in \overline{B}$, such that*

- $0 < v(t) < b/a$ *for all* $t \in [0, R[\,$;
- $v'(t) < 0$ *for all* $t \in \,]0, R]$;
- $v''(t) < 0$ *for all* $t \in [0, R]$.

Singular solutions on thick shells

Let $S = S_{r,R}(x_0) = \{x \in \mathbb{R}^N \mid r < |x - x_0| < R\}$ be the spherical shell centered at x_0 and having radii r, R, with $0 < r < R$.

Theorem 2.2. *For any given $N \geq 2$, $a > 0$ and $r > 0$, there exist $R^* > 0$ and $b^* > 0$ such that, for all $R > R^*$ and $b > b^*$, there is a unique generalized solution u of* (1.1)*, with $\Omega = S$, which is radially symmetric, singular and satisfies*

$$
\begin{gathered}
u \in C^2(S \cup \partial B), \quad u(x) = 0 \ \text{ if } |x - x_0| = R, \\
u(x) > 0 \ \text{ if } \Big[\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nu\Big](x) = -1 \ \text{ if } |x - x_0| = r.
\end{gathered}
$$

Classical solutions on thin shells

It is worth observing that the conclusions of Theorem 2.2 fail if R is not bounded away from r.

Theorem 2.3. *For any given $N \geq 2$, $a > 0$, $b > 0$ and $r > 0$, there exists $R_* > 0$ such that, for all $R \in \,]r, R_*[\,$, there is a unique generalized solution u of* (1.1)*, with $\Omega = S$, which is radially symmetric and classical, with $u \in C^2(\overline{S})$.*

3 Small classical solutions on arbitrary domains

If Ω is an arbitrary bounded regular domain in $\mathbb{R}^N$, the existence of a maximal connected two-dimensional branch of classical solutions, which emanates from the line of trivial solutions, can be established.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in \,]0,1[\,$. Then, there exists a set*

$$
\mathcal{S} = \bigcup_{a>0} \big(\{a\} \times [0, b_\infty(a)[\big) \subseteq \mathbb{R}_0^+ \times \mathbb{R}^+
$$

such that, for any $(a,b) \in \mathcal{S} \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$, problem (1.1) *has a unique generalized solution $u = u(a,b) \in C^{2,\alpha}(\overline{\Omega})$, which is classical, asymptotically stable, smoothly depends on the parameters (a,b) in the topology of $C^{2,\alpha}(\overline{\Omega})$, and satisfies, for every $a > 0$,*

$$
\lim_{b \to 0} \|u(a,b)\|_{C^{2,\alpha}} = 0
$$

and, in case $b_\infty(a) < +\infty$,

$$
\limsup_{b \to b_\infty(a)} \|\nabla u(a,b)\|_\infty = +\infty.
$$

4 Generalized solutions on arbitrary domains

The proof of the existence of generalized solutions is conceptually delicate and technically elaborate. It requires the study, in the space of bounded variation functions, of a suitable action functional, involving an anisotropic area term, whose minimizers give raise, via a change of variables, to the generalized solutions. The interior regularity of these bounded variation minimizers is obtained by combining a delicate approximation scheme with a "local" existence result basically due to Serrin [18] and the classical gradient estimates of Ladyzhenskaya and Ural'tseva [8].

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$. Then, for every $a > 0$, $b > 0$, there exists a unique generalized solution u of problem* (1.1)*, which also satisfies:*

- *$u \in C^\infty(\Omega)$;*
- *the set of points $x_0 \in \partial\Omega$, where u is continuous and satisfies $u(x_0) = 0$, is non-empty;*
- *$0 < u(x) < b/a$ for all $x \in \Omega$;*
- *u minimizes in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ the functional*

$$\int\limits_{\Omega} e^{-bz}\sqrt{1+|\nabla z|^2}\,dx - \frac{a}{b}\int\limits_{\Omega} e^{-bz}\Big(z+\frac{1}{b}\Big)\,dx + \frac{1}{b}\int\limits_{\partial\Omega} |e^{-bz}-1|\,d\mathcal{H}^{N-1}.$$

Remarks. The second conclusion of Theorem 4.1 can be further specified as follows: u is continuous at x_0 and satisfies $u(x_0) = 0$ at any point $x_0 \in \partial\Omega$ where an exterior sphere condition holds with radius $r \geq (N-1)\,b/a$ (i.e., there exists a point $y \in \mathbb{R}^N$ such that the open ball $B(y,r)$ of center y and radius r satisfies $B(y,r) \cap \Omega = \varnothing$ and $x_0 \in \overline{B(y,r)} \cap \partial\Omega$). Clearly, an exterior sphere condition, with arbitrary radius, holds at all points $x_0 \in \partial\Omega$ belonging to the boundary of the convex hull of $\overline{\Omega}$. The last conclusion of Theorem 4.1 also shows that all generalized solutions of (1.1) enjoy some form of stability.

5 Classical versus singular solutions

Combining the previous results yields a rather complete picture of the structure of the solution set of problem (1.1).

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in]0,1[$. Then, for every $a > 0$, either for all $b > 0$ problem* (1.1) *has a unique generalized solution, which is classical, or there exists $b^* = b^*(a) \in]0,+\infty[$ such that*

- *if $b \in]0,b^*]$, then problem* (1.1) *has a unique generalized solution u, which is classical;*
- *if $b \in]b^*,+\infty[$, then problem* (1.1) *has a unique generalized solution u, which is singular.*

In addition, the following conclusions hold:

- *the map $a \longmapsto b^*(a)$ is non-decreasing, with $\inf\limits_{a>0} b^*(a) > 0$;*
- *the map $(a,b) \longmapsto u(a,b)$ is continuous from $\mathbb{R}^+_0 \times \mathbb{R}^+$ to $L^\infty(\Omega)$;*
- *for any $a > 0$, the map $b \longmapsto u(a,b)$ is increasing in the sense that if $b_1 < b_2$, then $u(a,b_1) < u(a,b_2)$ in Ω;*
- *for any $b > 0$, the map $a \longmapsto u(a,b)$ is decreasing in the sense that if $a_1 < a_2$, then $u(a_1,b) > u(a_2,b)$ in Ω.*

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