The Asymptotic Properties of Slowly Varying Solutions of Second Order Differential Equations with Regularly and Rapidly Varying Nonlinearities

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The aim of the work is to find necessary and sufficient conditions of existence of sufficiently wide special class of solutions of second order differential equations with regularly and rapidly varying nonlinearities and to obtain asymptotic representations for such solutions and their derivatives of the first order.

Second order differential equations with power and exponential nonlinearities play an important role in development of the qualitative theory of differential equations. Such equations also have a lot of applications in practice. It happens, for example, when we study the distribution of electrostatic potential in a cylindrical volume of plasma of products of burning.

The corresponding equation may be reduced to the following one:

$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$

In the work of V. M. Evtuhov and N. G. Drik [3], some results on asymptotic behavior of solutions of such equations have been obtained.

Exponential nonlinearities form a special class of rapidly varying nonlinearities. The consideration of the last ones is necessary for some models. All this makes the topic of our research actual.

Our investigations need establishment of the next class of functions.

We call the measurable function $\varphi : \Delta_Y \to]0, +\infty[$ a regularly varying as $y \to Y$, $z \in \Delta_Y$ of index σ [1] if for every $\lambda > 0$ we have

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{\varphi(\lambda y)}{\varphi(y)} = \lambda^{\sigma}.$$

Here $Y \in \{0, \pm\infty\}$, Δ_Y is some one-sided neighbourhood of Y. If $\sigma = 0$, such function is called slowly varying.

The function $\varphi : [s, +\infty[\to]0, +\infty[$ $(s > 0)$ is called a rapidly varying function [1] of the $+\infty$ order on infinity if this function is measurable and

$$\lim_{y \to \infty} \frac{\varphi(\lambda y)}{\varphi(y)} = \begin{cases} 0 & \text{at } 0 < \lambda < 1, \\ 1 & \text{if } \lambda = 1, \\ +\infty & \text{at } \lambda > 1. \end{cases}$$

It is called a rapidly varying function of the $-\infty$ order on infinity if

$$\lim_{y \to \infty} \frac{\varphi(\lambda y)}{\varphi(y)} = \begin{cases} +\infty & \text{if } 0 < \lambda < 1, \\ 1 & \text{at } \lambda = 1, \\ 0 & \text{if } \lambda > 1. \end{cases}$$

The function $\varphi(y)$ is called a rapidly varying function of zero order if $\varphi(\frac{1}{y})$ is a rapidly varying function of $+\infty$ order. An exponential function is a special case of the last ones.

The differential equation

$$y'' = \alpha_0 p(t)\varphi(y),$$

with a rapidly varying function φ, was investigated in the work of V. M. Evtuhov and V. M. Kharkov [4]. But in the mentioned work the introduced class of solutions of the equation depends on the function φ. This is not convenient for practice.

The more general class of equations of such type is established in this work.

Let us consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'), \tag{1}$$

where $\alpha_0 \in \{-1;1\}$, $p : [a,\omega[\to]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ $(i \in \{0,1\})$ – are continuous functions, $Y_i \in \{0,\pm\infty\}$, Δ_{Y_i} – is one-sided neighborhood of Y_i.

Furthermore, we assume that function φ_1 is a regularly varying function as $y \to Y_1$ $(y \in \Delta_{Y_1})$ of the order σ_1, and function φ_0 is twice continuously differentiable and satisfies the following limit relations

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi_0(y) \in \{0,+\infty\}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

From conditions (2) it can be proved that φ_0 and its derivatives of the first order are rapidly varying function as $y \to Y_0$ $(y \in \Delta_{Y_0})$.

The main aim of our research is the development of methods of establishing asymptotic representations of solutions of such differential equations in order to receive a new class of mentioned equations.

We use a lot of methods of mathematical analysis, linear algebra, analytic geometry, theory of homogeneous differential equations in our work. Some special methods of investigation of equations of the mentioned type, being developed by the superiors, are also used.

We call solution y of the equation (1) defined on $[t_0,\omega[\subset [a,\omega[$, the $P_\omega(Y_0,Y_1,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if the following conditions take place

$$y^{(i)} : [t,\omega[\to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

In this work we consider $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) in case $\lambda_0 = 0$. Because of the properties of these solutions (see, eg., [2]) all of them are slowly varying functions as $t \uparrow \omega$. Therefore the case $\lambda_0 = 0$ is one of the most difficult for research. The problem of investigation $P_\omega(Y_0,Y_1,0)$-solutions for equations with rapidly varying functions is difficult by the fact that composition of rapidly and regularly varying functions may be as rapidly, as regularly, as slowly varying function as the argument tents to the singular point.

We have obtained the necessary and sufficient conditions for the existence of $P_\omega(Y_0,Y_1,0)$-solutions of equation (1) and find asymptotic representations of these solutions and their derivatives of the first order.

Now we need the following notations

$$\pi_\omega(t)=\begin{cases} t & \text{if } \ \omega=+\infty,\\ t-\omega & \text{if } \ \omega<+\infty, \end{cases} \quad \theta_1(y)=\varphi_1(y)|y|^{-\sigma_1},$$

$$I(t)=\operatorname{sign}(y_1^0)\times\int_{B_\omega^0}^{t}\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{y_1^0}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^0=\begin{cases} b & \text{if } \displaystyle\int_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{y_1^0}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{y_1^0}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty, \end{cases}$$

$$\Phi_0(y)=\int_{A_\omega^0}^{y}|\varphi_0(s)|^{\frac{1}{\sigma_1-1}}\,ds,\quad A_\omega^0=\begin{cases} y_0^0 & \text{if } \displaystyle\int_{y_0^0}^{Y_0}|\varphi_0(y)|^{\frac{1}{\sigma_1-1}}\,dy=+\infty,\\ Y_0 & \text{if } \displaystyle\int_{y_0^0}^{Y_0}|\varphi_0(y)|^{\frac{1}{\sigma_1-1}}\,dy<+\infty, \end{cases}$$

$$\operatorname{sign}\varphi_0(y)=f_1 \ \text{ as } \ y\in\Delta_{Y_0},\quad Z_1=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y).$$

The inferior limits of the integrals are chosen in such forms that the corresponding integrals tend either to 0 or to ∞ as $t\uparrow\omega$ and $y\to Y_0$, $y\in\Delta_{Y_0}$, correspondingly.

Note some necessary definitions.

Definition 1. Let $Y\in\{0,\pm\infty\}$, Δ_Y – is some one-sided neighborhood of Y. The continuously differentiable function $L:\Delta_Y\to]0,+\infty[$ is called normaliyed slowly varying function [5] as $y\to Y$ ($y\in\Delta_Y$) if

$$\lim_{\substack{y\to Y_1\\ y\in\Delta_{Y_i}}}\frac{yL'(y)}{L(y)}=0.$$

Definition 2. We say that a slowly varying as $y\to Y$ ($y\in\Delta_Y$) function $\theta:\Delta_Y\to]0,+\infty[$ satisfies the condition S if for any normaliyed slowly varying function $L:\Delta_{Y_i}\to]0,+\infty[$ the following condition takes place

$$\theta(yL(y))=\theta(y)(1+o(1)) \ \text{ as } \ y\to Y \ \ (y\in\Delta_Y).$$

Remark 1. The following statement is true

$$\Phi(y)=(\sigma_1-1)\,\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(y)}{\varphi_0'(y)}\,[1+o(1)] \ \text{ as } \ y\to Y_0 \ \ (y\in\Delta_{Y_0}).$$

From this as $y\in\Delta_{Y_0}$, we have

$$\operatorname{sign}(\varphi_0'(y)\Phi(y))=\operatorname{sign}(\sigma_1-1).$$

Remark 2. Because of conditions (2) on the function φ_0, we have that $z_1 \in \{0, +\infty\}$ and

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\Phi''(y) \cdot \Phi(y)}{(\Phi'(y))^2} = 1.$$

The following conclusions take place for equation (1).

Theorem 1. *Let $\sigma_1 \neq 1$. Then for the existence of $P_\omega(Y_0, Y_1, 0)$-solutions of the equation* (1) *such that the following finite or infinite limit exists*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) y''(t)}{y'(t)},$$

it's necessary the following conditions

$$f_1 I(t)(\sigma_1 - 1) > 0, \quad \alpha_0 \pi_\omega(t) y_1^0 < 0 \ \text{ as } \ t \in [a, \omega[\,, \tag{3}$$

$$\lim_{t \uparrow \omega} \frac{y_1^0}{|\pi_\omega(t)|} = Y_1, \quad \lim_{t \uparrow \omega} I(t) = Z_1, \quad \lim_{t \uparrow \omega} \frac{I'(t) \pi_\omega(t)}{\Phi'(\Phi^{-1}(I(t)))\Phi^{-1}(I(t))} = 0 \tag{4}$$

to be fulfilled.

If the function θ_1 satisfies the condition S, the following finite or infinite limit exists $\lim\limits_{t \uparrow \omega} \frac{\pi_\omega(t) I'(t)}{I(t)}$, the function $\frac{\pi_\omega(t) \cdot I'(t)}{I(t)}$ is a normalized slowly varying function as $t \uparrow \omega$, the function $(\frac{\Phi'(y)}{\Phi(y)})$ is a regularly varying function of the order γ_0 as $y \to Y_0$ ($y \in \Delta_{Y_0}$), $(\gamma_0 + 1) < 0$ as $Y_0 = 0$, and $(\gamma_0 + 1) > 0$ in another case, and

$$\lim_{t \uparrow \omega} \left| \frac{\pi_\omega(t) I'(t)}{I(t)} \right| < +\infty$$

or

$$\pi_\omega(t) \cdot I(t) \cdot I'(t)(1 - \sigma_1) > 0, \ \text{ when } \ t \in [a, \omega[\,,$$

then (3), (4) *are sufficient conditions for the existence of such solutions for the equation* (1). *For every $P_\omega(Y_0, Y_1, 0)$-solution the following asymptotic representations take place as $t \uparrow \omega$*

$$\Phi(y(t)) = I(t)[1 + o(1)], \quad \frac{y'(t)\Phi'(y(t))}{\Phi(y(t))} = \frac{I'(t)}{I(t)}\,[1 + o(1)].$$

References

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