On Baire Classes of Lyapunov Invariants

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For a given $n \in \mathbb{N}$ let us denote by $\mathcal{M}^n$ the set of linear systems of the form

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+ \equiv [0, +\infty), \tag{1}$$

where A is a piecewise continuous matrix function (which we identify with the respective system) and by $\widehat{\mathcal{M}}^n$ the subset of $\mathcal{M}^n$ comprising systems with bounded coefficients.

The set $\mathcal{M}^n$ is endowed with the *uniform* and *compact-open* topologies defined respectively by the metrics

$$\rho_U(A,B) = \sup_{t \in \mathbb{R}^+} \min \big\{ \|A(t) - B(t)\|, 1 \big\}, \quad \rho_C(A,B) = \sup_{t \in \mathbb{R}^+} \min \big\{ \|A(t) - B(t)\|, 2^{-t} \big\},$$

with $\|\cdot\|$ being a matrix norm (e.g., the spectral one). The resulting topological spaces will be denoted by $\mathcal{M}^n_U$ and $\mathcal{M}^n_C$. Similar notation will be used for their subspaces.

As early as 1928, O. Perron [9] (see also [4, 1.4]) discovered that for $n \ge 2$ the largest Lyapunov exponent is not upper semi-continuous as a functional on the space $\widehat{\mathcal{M}}^n_U$. He also suggested sufficient conditions for a system (1) to be a point of continuity of all the Lyapunov exponents in the uniform topology, which is commonly used in the study of the effect of perturbations on one or the other property of a system.

Further development of the theory of linear systems has led to introduction of a whole range of asymptotic behaviour characteristics, many of which proved to be discontinuous with respect to the uniform topology.

In a seminal work [7] V. M. Millionshchikov proposed using the Baire classification of functions to describe the dependence of those characteristics on the system coefficients. Motivated by parametric families of systems, V. M. Millionshchikov actively studied the compact-open topology on $\mathcal{M}^n$ and systematically tried to get rid of the assumption that the coefficients of (1) are bounded.

Let us introduce a piece of useful notation. Let M be a metric space and F be a set of functions $f : M \to \overline{\mathbb{R}}$. Define for each countable ordinal α the set $[F]_\alpha$ by transfinite induction as follows:

1) $[F]_0 = F$;

2) $[F]_\alpha$ is the set of functions $f : M \to \overline{\mathbb{R}}$ representable in the form

$$f(x) = \lim_{k \to \infty} f_k(x), \ \ x \in M,$$

where functions f_k, $k \in \mathbb{N}$, belong to the sets $[F]_\xi$ with $\xi < \alpha$.

Definition 1 ([5, § 31.IX])**.** Let M be a metric space and α be a countable ordinal. The α-th *Baire class* $\mathfrak{F}_\alpha(M)$ is defined by $\mathfrak{F}_\alpha(M) = [C(M)]_\alpha$, $C(M)$ being the set of continuous functions $f : M \to \overline{\mathbb{R}}$. The class $\mathfrak{F}^0_\alpha(M) = \mathfrak{F}_\alpha(M) \setminus \bigcup\limits_{\xi<\alpha} \mathfrak{F}_\xi(M)$ is called the α-th *exact* Baire class. For convenience, let us denote by $\mathfrak{F}^0_{\omega_1}(M)$ the set of functions which do not belong to any of the classes $\mathfrak{F}_\alpha(M)$, $\alpha \in [0, \omega_1)$ (here and subsequently, ω_1 is the first uncountable ordinal).

V. M. Millionshchikov proved [8] that the Lyapunov exponents belong to the class $\mathfrak{F}_2(\mathcal{M}^n_C) \subset \mathfrak{F}_2(\mathcal{M}^n_U)$. Later M. I. Rakhimberdiev [10] proved that for $n \geq 2$ they do not belong to the class $\mathfrak{F}_1(\widehat{\mathcal{M}}^n_U) \supset \mathfrak{F}_1(\widehat{\mathcal{M}}^n_C)$. Therefore, for $n \geq 2$ the Lyapunov exponents (and their restrictions to $\widehat{\mathcal{M}}^n$) belong to the second exact Baire classes on both spaces $\mathcal{M}^n_C$ and $\mathcal{M}^n_U$ ($\widehat{\mathcal{M}}^n_C$ and $\widehat{\mathcal{M}}^n_U$, respectively).

Investigations in this vein have been continued by V. M. Millionshchikov himself, his students and followers. It was established by efforts of several authors [2, 11] that the minorants of the Lyapunov exponents belong to the class $\mathfrak{F}_3(\widehat{\mathcal{M}}^n_C)$, and A. N. Vetokhin proved [14] that they do not belong to the class $\mathfrak{F}_2(\widehat{\mathcal{M}}^n_C)$. Thus they belong to the third exact class on the space $\widehat{\mathcal{M}}^n_C$ (at the same time, they are known to belong to the first exact class on the space $\widehat{\mathcal{M}}^n_U$).

The natural question arises: for which $\alpha, \beta, \gamma, \delta \in [0, \omega_1]$ there exists an asymptotic invariant [1] from $\mathfrak{F}^0_\gamma(\mathcal{M}^n_U) \cap \mathfrak{F}^0_\delta(\mathcal{M}^n_C)$ such that its restriction to $\widehat{\mathcal{M}}^n$ belongs to $\mathfrak{F}^0_\alpha(\widehat{\mathcal{M}}^n_U) \cap \mathfrak{F}^0_\beta(\widehat{\mathcal{M}}^n_C)$?

Let us make the notion of asymptotic invariant more precise for the purposes of this paper (see the discussion of this notion in [6, § 2]).

Definition 2 ([3, Chapter IV, § 2])**.** Systems $A, B \in \mathcal{M}^n$ are said to be *weakly Lyapunov equivalent* if they possess fundamental matrices $X(\cdot)$ and $Y(\cdot)$ such that

$$\sup_{t\in\mathbb{R}^+} \big(\|X(t)Y^{-1}(t)\| + \|Y(t)X^{-1}(t)\|\big) < \infty.$$

A functional taking equal values at any weakly Lyapunov equivalent systems is called a *weak Lyapunov invariant.*

Proposition 1 ([13])**.** *Classes $\mathfrak{F}^0_1(\mathcal{M}^n_C)$ and $\mathfrak{F}^0_1(\widehat{\mathcal{M}}^n_C)$ do not contain any weak Lyapunov invariants.*

Let us note that the index of the exact Baire class of a function on a space is not less than that of its restriction to a subspace and also that the index of the exact Baire class of a function on $\mathcal{M}^n_C$ is not less than that on $\mathcal{M}^n_U$ (since the uniform topology is finer).

The following theorem states that a quadruple of the indices of the exact Baire classes with respect to the compact-open and uniform topologies containing a weak Lyapunov invariant and its restriction to $\widehat{\mathcal{M}}^n$ is subject to no restrictions except the natural ones mentioned above and those implied by Proposition 1.

Theorem 1. *Let ordinals $\alpha, \beta, \gamma, \delta \in [0, \omega_1]$ be given. Then a weak Lyapunov invariant satisfying the conditions*

1) $\varphi \in \mathfrak{F}^0_\gamma(\mathcal{M}^n_U) \cap \mathfrak{F}^0_\delta(\mathcal{M}^n_C)$;

2) $\varphi\big|_{\widehat{\mathcal{M}}^n} \in \mathfrak{F}^0_\alpha(\widehat{\mathcal{M}}^n_U) \cap \mathfrak{F}^0_\beta(\widehat{\mathcal{M}}^n_C)$,

exists if and only if

$$\alpha \leq \min\{\beta, \gamma\}, \quad \max\{\beta, \gamma\} \leq \delta, \ \ \beta \neq 1, \ \ \delta \neq 1.$$

Definition 3 ([12])**.** Let $\mathcal{M} \subset \mathcal{M}^n$. We say that a functional $\varphi : \mathcal{M} \to \overline{\mathbb{R}}$ *has a compact support* if there exists $T > 0$ such that $\varphi(A) = \varphi(B)$ whenever $A, B \in \mathcal{M}$ coincide on the interval $[0, T]$. The set of all functionals on $\mathcal{M}$ with compact support is denoted by $\mathfrak{C}(\mathcal{M})$.

Remark 1. In the abstract [12] functionals with compact support are called boundedly dependent.

Suppose that a functional defined on a subspace of $\mathcal{M}^n_C$ is the repeated pointwise limit of a sequence of continuous ones. As noted in [12], the desire to compute the values of those based only on information on the system on finite time intervals naturally leads to the requirement that their supports be compact.

Definition 4. Let $\mathcal{M} \subset \mathcal{M}^n_C$. Define the α-th *formula class* $\mathfrak{C}_\alpha(\mathcal{M})$ by (cf. [12])

$$\mathfrak{C}_\alpha(\mathcal{M}) = [\mathfrak{F}_0(\mathcal{M}) \cap \mathfrak{C}(\mathcal{M})]_\alpha, \ \ \alpha \in [0, \omega_1).$$

Proposition 2 ([12])**.** *Let $\mathcal{M} \subset \mathcal{M}^n_C$. Then $\mathfrak{C}_\alpha(\mathcal{M}) \subset \mathfrak{F}_\alpha(\mathcal{M}) \subset \mathfrak{C}_{\alpha+1}(\mathcal{M})$ for all $\alpha \in [0, \omega_1)$. Moreover, for $\mathcal{M} = \mathcal{M}^n_C$ and $\alpha = 0$ the first inclusion is strict.*

Let a functional defined on a subspace of $\mathcal{M}^n_C$ be the repeated limit of a sequence of continuous ones. The next theorem states that the latter could be chosen to have compact support.

Theorem 2. *Let $\mathcal{M} \subset \mathcal{M}^n_C$. Then $\mathfrak{C}_\alpha(\mathcal{M}) = \mathfrak{F}_\alpha(\mathcal{M})$ for all $\alpha \in [1, \omega_1)$.*

The case $\alpha = 0$ is totally different as the next theorem shows.

Theorem 3. *Let $\mathcal{M} \subset \mathcal{M}^n_C$. Then $\mathfrak{C}_0(\mathcal{M}) = \mathfrak{F}_0(\mathcal{M})$ if and only if there exists $T > 0$ such that $A = B$ whenever $A, B \in \mathcal{M}$ coincide on the interval $[0, T]$.*

It appears that, generally speaking, one cannot decrease the number of limits in a formula for a weak Lyapunov invariant by allowing the prelimit functionals with compact support to be discontinuous.

Theorem 4. *Let $\mathcal{M} \supset \{A \in \mathcal{M}^n : \sup\limits_{t \geq 0} \|A(t)\| \leq 1\}$ be endowed with the compact-open topology. Then for all $\alpha \in [1, \omega_1)$ there exists a weak Lyapunov invariant $\varphi \in \mathfrak{F}_{\alpha+1}(\mathcal{M}) \setminus [\mathfrak{C}(\mathcal{M})]_\alpha$.*

For $\alpha = 1$ the statement of the above theorem can be strengthened: no nontrivial weak Lyapunov invariant is the limit of a sequence of functionals with compact support.

Theorem 5. *If $\mathcal{M} \in \{\widehat{\mathcal{M}^n_C}, \mathcal{M}^n_C\}$, then $[\mathfrak{C}(\mathcal{M})]_1$ does not contain weak Lyapunov invariants except constants.*

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