On a Four-Point Boundary Value Problem for Second Order Linear Functional Differential Equations

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The multi-point and nonlocal boundary value problems for ordinary and functional differential equations have been studied by many authors in recent years, see [1–20] and references therein. Nonlocal boundary value problems arise in many applications and can be used for modeling [2, 9, 11, 18].

In the resonance and non-resonance cases, many authors (see, for instance, [2, 3, 5, 6, 10–12, 14, 15, 18, 20]) consider, firstly, the boundary value problem for a linear ordinary differential equation. They established the existence of a unique solution, investigate the properties of the Green function, then apply the results to non-linear equations.

Motivated by the above work, in this paper, we consider a four-point boundary value problem for linear second order functional differential equation at resonance. We obtain sharp sufficient conditions for the existence and uniqueness of solutions. So, the results of many previous works on multi-point boundary value problems can be extended in the case of this four-point problem.

Let us define some sets and functions:

$$\Omega \equiv \big\{(b,c):\ 0\le b\le c\le 1\big\},\quad \Omega_1 \equiv \Big\{(b,c)\in\Omega:\ c\ge 3b-1,\ c\ge \frac{b+1}{3}\Big\},$$
$$\Omega_2 \equiv \Big\{(b,c)\in\Omega:\ c<\frac{b+1}{3}\Big\},\quad \Omega_3\equiv\{(b,c)\in\Omega: c<3b-1\}$$

(it is clear that $\Omega_1\cup\Omega_2\cup\Omega_3=\Omega$),

$$d_2(b,c)\equiv\sqrt{(3b-1-c)(1+c-b)},\quad d_3(b,c)\equiv\sqrt{(1+b-3c)(1+c-b)},$$
$$\omega_2(b,c)\equiv\Big[\frac{b-d_2(b,c)}{2},\frac{b+d_2(b,c)}{2}\Big],\quad \omega_3(b,c)\equiv\Big[\frac{1+c-d_3(b,c)}{2},\frac{1+c+d_3(b,c)}{2}\Big],$$
$$h_2(b,c,t)\equiv\frac{2}{t^2}\Big(\frac{b(1+c-b)}{((1+c)/2-t)^2}-1\Big),\quad t\in\omega_2,$$
$$h_3(b,c,t)\equiv\frac{2}{(1-t)^2}\Big(\frac{(1-c)(1+c-b)}{(t-b/2)^2}-1\Big),\quad t\in\omega_3.$$

Let

$$M(b,c)\equiv\begin{cases}\dfrac{32}{(1+c-b)^2} & \text{if } (b,c)\in\Omega_1;\\ \min\limits_{t\in\omega_2(b,c)} h_2(b,c,t) & \text{if } (b,c)\in\Omega_2;\\ \min\limits_{t\in\omega_3(b,c)} h_3(b,c,t) & \text{if } (b,c)\in\Omega_3.\end{cases}$$

Definition. A linear operator T from the space of all continuous real functions $\mathbf{C}[0,1]$ into the space of all integrable functions $\mathbf{L}[0,1]$ is called positive if it maps every nonnegative continuous function into an almost everywhere nonnegative integrable function.

Theorem 1. *Let $0 < b \le c < 1$, $p \in \mathbf{L}[0,1]$ be a non-negative function, $h : [0,1] \to [0,1]$ be a measurable function.*

Then the boundary value problem

$$\begin{cases} \ddot{x}(t) = p(t)x(h(t)) + f(t), & t \in [0,1], \\ x(0) = x(b), \quad x(c) = x(1), \end{cases} \tag{1}$$

has a unique solution for every $f \in \mathbf{L}[0,1]$ if

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le M(b,c), \ \ p \not\equiv 0, \ \ p \not\equiv M(b,c).$$

Remark. The constant $M(b,c)$ is the best one. If $p(t) \equiv P > M(b,c)$, then there exists a measurable function $h : [0,1] \to [0,1]$ such that problem (1) has no a unique solution.

Theorem 1 can be transferred to a more general case.

Theorem 2. *Let $0 < b \le c < 1$, $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*

Then the boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), \ \ t \in [0,1], \\ x(0) = x(b), \quad x(c) = x(1), \end{cases} \tag{2}$$

has a unique solution for every $f \in \mathbf{L}[0,1]$ if

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le M, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv M.$$

We can get some simple corollaries about the solvability of problem (2) for different b and c satisfying the condition $0 < b \le c < 1$. The cases $b = 0$ or $c = 1$ correspond to the boundary value conditions $\dot{x}(0) = 0$ and $\dot{x}(1) = 0$. These cases can be dealt by the similar way.

Corollary 1. *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*

Then the boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ x(0) = x\Big(\frac{1}{2}\Big) = x(1), \end{cases}$$

has a unique solution for every $f \in \mathbf{L}[0,1]$ if

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le 32, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv 32.$$

Corollary 2. *Let $b \in (0, 1/2)$, $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*

Then the boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ x(0) = x(b), \quad x(1-b) = x(1), \end{cases}$$

has a unique solution for every $f \in \mathbf{L}[0,1]$ if

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le \frac{8}{(1-b)^2}\,, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv \frac{8}{(1-b)^2}\,.$$

Corollary 3. *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*
Then the boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ \dot{x}(0) = 0, \quad x(0) = x(1) \quad (\text{or } \dot{x}(1) = 0, \quad x(0) = x(1)), \end{cases}$$

has a unique solution for every $f \in \mathbf{L}[0,1]$ if

$$\operatorname*{vrai\,sup}_{t\in[0,1]}(T1)(t) \leq 11 + 5\sqrt{5}, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv 11 + 5\sqrt{5}.$$

Corollary 4. *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear positive operator.*
Then the boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1], \\ \dot{x}(0) = 0, \quad \dot{x}(1) = 0, \end{cases}$$

has a unique solution for every $f \in \mathbf{L}[0,1]$ if

$$\operatorname*{vrai\,sup}_{t\in[0,1]}(T1)(t) \leq 8, \ \ T1 \not\equiv 0, \ \ T1 \not\equiv 8.$$

The constants in Theorem 2 and all corollaries are sharp.

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