Asymptotic Behavior of Some Special Classes of Solutions of Essentially Nonlinear n-th Order Differential Equations

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The following differential equation

$$y^{(n)} = \alpha_0 p(t) \exp\big(R(|\ln|y^{(n-1)}||)\big) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}) \tag{1}$$

is considered. In (1) $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\,^1 \to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, \dots, n)$ are continuous functions, $R :]0, +\infty[\to]0, +\infty[$ is continuously differentiable function, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[\,^2$, or the interval $]Y_i, y_i^0]$. We suppose also that R is a regularly varying on infinity function of index μ, $0 < \mu < 1$, every $\varphi_i(z)$ is a regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $\sum\limits_{i=0}^{n-1} \sigma_i \neq 1$.

We call the measurable function $\varphi : \Delta_Y \to]0, +\infty[$ a regularly varying as $z \to Y$ of index σ if for every $\lambda > 0$ we have

$$\lim_{\substack{z \to Y \\ z \in \Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma,$$

where $Y \in \{0, \pm\infty\}$, Δ_Y is some one-sided neighbourhood of Y. If $\sigma = 0$, such function is called a slowly varying.

It follows from the results of monograph [5] that regularly varying functions have the following properties.

M_1: Function $\varphi(z)$ is regularly varying of index σ as $z \to Y$ if and only if it admits the representation

$$\varphi(z) = z^\sigma \theta(z),$$

where $\theta(z)$ is a slowly varying function as $z \to Y$.

M_2: If function $L : \Delta_{Y^0} \to]0, +\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y^0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to]0, +\infty[$ is slowly varying as $z \to Y$.

M_3: If function $\varphi : \Delta_Y \to]0, +\infty[$ satisfies the condition

$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{z\varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$

then φ is regularly varying as $z \to Y$ of index σ.

[1] If $\omega > 0$, we take $a > 0$.
[2] If $Y_i = +\infty$ $(Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0;+\infty[$ satisfies the condition S if for any continuous differentiable function $L : \Delta_{Y_i} \to]0;+\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the following condition takes place

$$\theta(zL(z)) = \theta(z)(1+o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

We call defined on $[t_0,\omega[\subset [a,\omega[$ solution y of the equation (1) the $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if the following conditions take place

$$y^{(i)} : [t_0,\omega[\to \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,\dots,n-1), \quad \lim_{t\uparrow\omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0.$$

In regular cases $\lambda_{n-1}^0 \in R\setminus\{0,\frac12,\frac23,\dots,\frac{n-2}{n-1}\}$, the $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solutions of the equation (1) have been established in [3]. Such $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solutions are regularly varying functions as $t\uparrow\omega$ of indexes different from $\{0,1,\dots,n-1\}$.

The cases $\lambda_0 \in \{0,\frac12,\frac23,\dots,\frac{n-2}{n-1}\}$ are singular. Such solutions are regularly varying functions as $t\uparrow\omega$ of indexes $\{0,1,\dots,n-1\}$, so such solutions or some of their derivatives are slowly varying functions as $t\uparrow\omega$. Therefore for investigation of such solutions we must put additional conditions on functions $\varphi_0,\dots,\varphi_{n-1}$ and on the function p. The case $\lambda_0=0$ is of the most difficult ones. It is presented in this work. The case was investigated before [1,4] only when $R(z)\equiv 1$ and the function $\varphi_{n-1}(z)|z|^{-\sigma_{n-1}}$ satisfies the condition S. For equations of type (1), that contain, for example, functions like $\exp(|\ln|y||^\mu)$ $(0<\mu<1)$, the asymptotic representations of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},0)$-solutions were not established before. Let us notice that function $\exp(R(|\ln|z||))$ does not satisfy the condition S.

Now we need the following subsidiary notations.

$$\gamma_0 = 1-\sum_{j=0}^{n-1}\sigma_j, \quad C=\frac{1}{1-\sigma_{n-1}}, \quad \eta=\prod_{j=0}^{n-3}((n-i-2)!)^{\sigma_i}, \quad \gamma=\sum_{i=0}^{n-3}(i+2-n)\sigma_i,$$

$$\theta_i(z)=\varphi_i(z)|z|^{-\sigma_i} \ \ (i=0,\dots,n-1),$$

$$Q(t) = -\pi_\omega(t)\Big|\frac{(1-\sigma_{n-1})}{\eta}|\pi_\omega(t)|^{-\gamma} I_0(t)\theta_{n-1}\big(y_{n-1}^0|I_0(t)|^{\frac{1}{1-\sigma_{n-1}}}\big)\Big|^{\frac{1}{1-\sigma_{n-1}}} \operatorname{sign} y_{n-1}^0,$$

$$I_0(t)=\int_{A_\omega^0}^t p(\tau)\,d\tau, \quad I_1(t)=\int_{A_\omega^1}^t \frac{Q(\tau)}{\pi_\omega(\tau)}\,d\tau,$$

$$A_\omega^0 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau)\,d\tau=+\infty,\\[2mm] \omega, & \text{if } \displaystyle\int_a^\omega p(\tau)\,d\tau<+\infty, \end{cases} \qquad A_\omega^1 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega \Big|\frac{Q(\tau)}{\pi_\omega(\tau)}\Big|\,d\tau=+\infty,\\[2mm] \omega, & \text{if } \displaystyle\int_a^\omega \Big|\frac{Q(\tau)}{\pi_\omega(\tau)}\Big|\,d\tau<+\infty. \end{cases}$$

The following conclusions take place.

Theorem 1. *Let in equation* (1) $\sigma_{n-1}\ne 1$, *the function* θ_{n-1} *satisfy the condition* S *and*

$$\lim_{t\uparrow\omega}\frac{R'(|\ln|I(t)||)I_1(t)I_0'(t)}{I_0(t)I_1'(t)} = 0.$$

We suppose also that there exists the finite or infinite limit

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{I_0(t)}\,. \tag{2}$$

Then the following conditions are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},0)$*-solutions of equation* (1),

$$\lim_{t\uparrow\omega}\frac{I_1'(t)I_0(t)}{p(t)I_1(t)}=0,\quad \lim_{t\uparrow\omega}y^0_{n-1}|I_0(t)|^{\frac{1}{1-\sigma_{n-1}}}=Y_{n-1},$$
$$\lim_{t\uparrow\omega}y^0_{n-2}|I_1(t)|^{\frac{1-\sigma_{n-1}}{\gamma_0}}=Y_{n-2},\quad \lim_{t\uparrow\omega}y^0_i|\pi_\omega(t)|^{n-i-2}=Y_i,$$
$$\alpha_0y^0_{n-1}(1-\sigma_{n-1})I_0(t)>0,\quad (1-\sigma_{n-1})\gamma_0y^0_{n-2}I_1(t)<0,$$
$$y^0_iy^0_{i+1}\pi_\omega(t)(n-i-2)>0\ \text{ as }\ t\in[a,\omega[\,.$$

Here $i=0,\dots,n-3$.

For any such solution the following asymptotic representations take place as $t\uparrow\omega$

$$\frac{y^{(n-1)}(t)}{\exp(R(|\ln|y^{(n-1)}(t)||))\prod\limits_{j=0}^{n-1}\varphi_j(y^{(j)}(t))}=\alpha_0(1-\sigma_{n-1})I_0(t)[1+o(1)], \tag{3}$$

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}=\frac{I_1'(t)(1-\sigma_{n-1})}{\gamma_0I_1(t)}\,[1+o(1)],\quad \frac{y^{(i)}(t)}{y^{(n-2)}(t)}=\frac{[\pi_\omega(t)]^{n-i-2}}{(n-i-2)!}\,[1+o(1)],\quad i=0,\dots,n-3. \tag{4}$$

Theorem 2. *Let in equation* (1) $\sigma_{n-1}\neq1$, *the function* θ_{n-1} *satisfy the condition* S *and*

$$\lim_{t\uparrow\omega}\frac{I_0(t)Q'(t)}{R'(|\ln|I(t)||)Q(t)I_0'(t)}=0.$$

We suppose also that there exists the finite or infinite limit (2). *Then the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0,Y_1,\dots,Y_{n-1},0)$*-solutions of equation* (1),

$$\lim_{t\uparrow\omega}\frac{I_0(t)}{p(t)R'(|\ln|I_0(t)||)}=0,\quad \lim_{t\uparrow\omega}y^0_{n-1}|I_0(t)|^{\frac{1}{1-\sigma_{n-1}}}=Y_{n-1},$$
$$\lim_{t\uparrow\omega}y^0_{n-2}\Big|\frac{Q(t)}{R'(|\ln|I_0(t)||)}\Big|^{\frac{1-\sigma_{n-1}}{\gamma_0}}=Y_{n-2},\quad \lim_{t\uparrow\omega}y^0_i|\pi_\omega(t)|^{n-i-2}=Y_i,$$
$$\alpha_0y^0_{n-1}(1-\sigma_{n-1})I_0(t)>0,\quad (1-\sigma_{n-1})\gamma_0Q(t)y^0_{n-2}y^0_{n-1}>0,$$
$$y^0_iy^0_{i+1}\pi_\omega(t)(n-i-2)>0\ \text{ as }\ t\in[a,\omega[\,.$$

Here $i=0,\dots,n-3$.

For any such solution the representation (3), *the second representation in* (4) *and the following asymptotic representation*

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}=\frac{I_1'(t)}{\gamma_0R'(|\frac{1-\sigma_{n-1}}{\ln|I_0(t)|}|)}\,[1+o(1)]$$

take place as $t\uparrow\omega$.

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