Fully Linearized Difference Scheme for Generalized Rosenau Equation

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We consider the generalized Rosenau equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\lambda\,\frac{\partial (u)^m}{\partial x}+\mu\,\frac{\partial^5 u}{\partial x^4\partial t}=0,\ \ (x,t)\in Q, \tag{1}$$

together with the initial and boundary conditions

$$u(x,0)=\varphi(x),\ \ x\in[a,b],\quad u(a,t)=u(b,t)=\frac{\partial^2 u(a,t)}{\partial x^2}=\frac{\partial^2 u(b,t)}{\partial x^2}=0,\ \ t\in[0,T]. \tag{2}$$

Here λ and μ are positive constants, $m\geq 2$ is a positive integer, and $Q=(a,b)\times(0,T)$.

In this article, two-level scheme is constructed to find the values of the unknown function on the first level, besides the term $\partial(u)^m/\partial x$ is approximated by the offered in [1] way. For the upper levels, as in [2], the known approximation are used for derivatives.

The domain $\overline{Q}$ is divided into rectangular grid by the points $(x_i,t_j)=(a+ih,j\tau)$, $i=0,1,2,\dots,n$, $j=0,1,\dots,J$, where $h=(b-a)/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

The value of mesh function U at the node (x_i,t_j) is denoted by U_i^j, that is $U_i^j=U(x_i,t_j)$.

We define the difference quotients (forward, backward, and central, respectively) in x and t directions as follows:

$$(U_i^j)_x:=\frac{U_{i+1}^j-U_i^j}{h},\quad (U_i^j)_{\overline{x}}:=\frac{U_i^j-U_{i-1}^j}{h},\quad (U_i^j)_{\overset{\circ}{x}}:=\frac{1}{2}\big((U_i^j)_x+(U_i^j)_{\overline{x}}\big),$$
$$(U_i^j)_t:=\frac{U_i^{j+1}-U_i^j}{\tau},\quad (U_i^j)_{\overline{t}}:=\frac{U_i^j-U_i^{j-1}}{\tau},\quad (U_i^j)_{\overset{\circ}{t}}:=\frac{1}{2}\big((U_i^j)_t+(U_i^j)_{\overline{t}}\big).$$

We approximate the problem (1), (2) by the difference scheme

$$(U_i^j)_{\overset{\circ}{t}}+\frac{1}{2}\,(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}+\frac{\lambda m}{2(m+1)}\,\Lambda U_i^j+\mu(U_i^j)_{\overline{x}x\overline{x}x\overset{\circ}{t}}=0, \tag{3}$$
$$i=1,2,\dots,n-1,\ \ j=1,2,\dots,J-1,$$

$$(U_i^0)_t+\frac{1}{2}\,(U_i^1+U_i^0)_{\overset{\circ}{x}}+\frac{\lambda m}{2(m+1)}\,\Lambda U_i^0+\mu(U_i^0)_{\overline{x}x\overline{x}xt}=0,\ \ i=1,2,\dots,n-1, \tag{4}$$

$$U_i^0=\varphi(x_i),\quad U_0^j=U_n^j=(U_0^j)_{\overline{x}x}=(U_n^j)_{\overline{x}x}=0\ \ i=0,1,\dots,n,\ \ j=0,1,\dots,n, \tag{5}$$

where

$$\Lambda U_i^j:=(U_i^j)^{m-1}(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}+\big((U_i^j)^{m-1}(U_i^{j+1}+U_i^{j-1})\big)_{\overset{\circ}{x}},\ \ j=1,2,\dots,J-1,$$
$$\Lambda U_i^0:=(U_i^0)^{m-1}(U_i^1+U_i^0)_{\overset{\circ}{x}}+\big((U_i^0)^{m-1}(U_i^1+U_i^0)\big)_{\overset{\circ}{x}},\ \ i=1,2,\dots,n-1.$$

The obtained algebraic equations are linear with respect to the values of unknown function for each new level.

An a priori estimate of a solution of the difference scheme (3)–(5) is obtained with the help of energy inequality method, from which follows a uniquely solvability of the scheme.

In the equality of the obtained discrete conservation law the initial energy depends explicitly only on initial data.

Stability and second order convergence of difference scheme is proved without any restriction on discretization parameters τ, h.

References

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[2] J. Hu and K. Zheng, Two conservative difference schemes for the generalized Rosenau equation. *Bound. Value Probl.* **2010**, Art. ID 543503, 18 pp.