# On the Baire Classes of the Sergeev Lower Frequency of Zeros, Signs, and Roots of Linear Differential Equations

**E. A. Barabanov and A. S. Vaidzelevich**

*Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus*
*E-mails:* bar@im.bas-net.by; voidelevich@gmail.com

For a given positive integer $n$, by $\widetilde{\mathcal{E}}^n$ we denote the set of linear homogeneous $n$th-order differential equations

$$y^{(n)}+a_1(t)y^{(n-1)}+\cdots+a_{n-1}(t)\dot{y}+a_n(t)y=0,\quad t\in\mathbb{R}_+\stackrel{\rm def}{=}[0,+\infty),\tag{1}$$

with continuous coefficients $a_i(\,\cdot\,)\colon\mathbb{R}_+\to\mathbb{R}$, $i=\overline{1,n}$. We identify the equation (1) and the row $a=a(\,\cdot\,)=(a_1(\,\cdot\,),\dots,a_n(\,\cdot\,))$ of its coefficients and hence denote the equation (1) by $a$ as well. By $S(a)$ we denote the solution set of the equation $a$, and by $S_*(a)$ we denote the set of its nonzero solutions, i.e. $S_*(a)=S(a)\setminus\{\mathbf{0}\}$.

Let $y(\,\cdot\,)$ be a real-valued function defined on some set $D\subset\mathbb{R}$. A point $t\in D$ is called a sign change point of a function $y(\,\cdot\,)$ if, in any neighborhood of that point, the function $y(\,\cdot\,)$ takes values of opposite signs. If $y(\,\cdot\,)$ is a continuous function, then a sign change point is its zero. If the function $y(\,\cdot\,)$ is defined in some neighborhood of its zero $t_0$, then the zero $t_0$ is referred to as a root of multiplicity $k$ of the function $y(\,\cdot\,)$ if at the point $t_0$ its first $k-1$ derivatives are zero and the $k$th derivative is nonzero.

Next, by $\varkappa$ we denote a symbol that takes values in the set of three elements $\{0,-,+\}$. For a function $y(\,\cdot\,)\colon\mathbb{R}_+\to\mathbb{R}$ and a number $t>0$, by $\nu^\varkappa(y(\,\cdot\,);t)$ we denote the following quantities for the function $y(\,\cdot\,)$ on the half-open interval $[0,t)$ depending on the value of $\varkappa$ : the number of zeros if $\varkappa=0$, the number of sign changes if $\varkappa=-$, and the sum of root multiplicities if $\varkappa=+$. If $t_0=0$ is a zero of the function $y(\,\cdot\,)$, then, for the computation of its multiplicity, all desired derivatives are considered to be right-sided. If the number of zeros or the number of sign changes or roots of the function $y(\,\cdot\,)$ on the half-open interval $[0,t)$ is infinite, then the corresponding values are considered to be equal to $+\infty$. It is easy to see that $\nu^\varkappa(y(\,\cdot\,);t)$ is a finite integer number for every symbol $\varkappa\in\{0,-,+\}$, nonzero solution $y(\,\cdot\,)$ of the equation (1), and $t>0$. Sergeev [7]– [9] introduced the following notion.

**Definition.** For any nonzero solution $y(\,\cdot\,)\in S_*(a)$ of the system $a$ the quantities

$$\hat{\nu}^{\varkappa}[y]\stackrel{\rm def}{=}\varlimsup_{t\to+\infty}\frac{\pi}{t}\,\nu^{\varkappa}(y(\,\cdot\,);t)\ \text{ and }\ \check{\nu}^{\varkappa}[y]\stackrel{\rm def}{=}\varliminf_{t\to+\infty}\frac{\pi}{t}\,\nu^{\varkappa}(y(\,\cdot\,);t)\tag{2}$$

are called the upper and lower characteristic frequencies, respectively, of zeros if $\varkappa=0$, signs if $\varkappa=-$, and roots if $\varkappa=+$.

Generally, the value of the quantities $\check{\nu}^{\varkappa}[y]$ and/or $\hat{\nu}^{\varkappa}[y]$ can be equal to $+\infty$. By $\overline{\mathbb{R}}$ we denote the extended numerical axis ($\overline{\mathbb{R}}=\mathbb{R}\sqcup\{-\infty,+\infty\}$) considered in the natural (ordinal) topology, and by $\overline{\mathbb{R}}_+$ we denote its nonnegative semiaxis.

For any $a\in\widetilde{\mathcal{E}}^n$, the asymptotic characteristics (2) define the mappings

$$\hat{\nu}^{\varkappa}[\,\cdot\,]\colon S_*(a)\to\overline{\mathbb{R}}_+\ \text{ and }\ \check{\nu}^{\varkappa}[\,\cdot\,]\colon S_*(a)\to\overline{\mathbb{R}}_+,\ \ \varkappa\in\{0,-,+\},\tag{3}$$

acting by the rules $y \longmapsto \hat{\nu}^{\varkappa}[y]$ and $y \longmapsto \check{\nu}^{\varkappa}[y]$, respectively. Instead of the mappings (3), it is more convenient to consider the functions $\hat{\nu}^{\varkappa}(\,\cdot\,)$ and $\check{\nu}^{\varkappa}(\,\cdot\,)$, $\varkappa \in \{0,-,+\}$, respectively, which are defined as follows. Since, between the vector space $S(a)$ of solutions of an equation $a \in \widetilde{\mathcal{E}}^n$ and the vector space $\mathbb{R}^n$, there is a natural isomorphism $\iota : S(a) \to \mathbb{R}^n$ acting by the rule $y(\,\cdot\,) \longmapsto (y(0), \dot{y}(0), \dots, y^{(n-1)}(0))^\top$, it follows that the mappings (3) define the functions

$$\hat{\nu}^{\varkappa}(\,\cdot\,) \stackrel{\text{def}}{=} \hat{\nu}^{\varkappa}[\,\cdot\,] \circ \iota^{-1} \colon \mathbb{R}^n_* \to \overline{\mathbb{R}}_+ \quad \text{and} \quad \check{\nu}^{\varkappa}(\,\cdot\,) \stackrel{\text{def}}{=} \check{\nu}^{\varkappa}[\,\cdot\,] \circ \iota^{-1} \colon \mathbb{R}^n_* \to \overline{\mathbb{R}}_+, \quad \varkappa \in \{0,-,+\}, \tag{4}$$

where $\mathbb{R}^n_* \stackrel{\text{def}}{=} \mathbb{R}^n \setminus \{\mathbf{0}\}$. Conversely, since $\iota$ is a bijection, one can see that the functions (4) define the mappings (3). The functions (4) have the following advantage in comparison with the mappings (3): the domains of those functions coincide for all equations from the set $\widetilde{\mathcal{E}}^n$.

Since the functions (4) (and the mappings (3)) are constant on any one-dimensional linear subspace with the excluded zero, it follows that, instead of the functions $\hat{\nu}^{\varkappa}(\,\cdot\,)$ and $\check{\nu}^{\varkappa}(\,\cdot\,)$, $\varkappa \in \{0,-,+\}$, one can consider their restrictions to the unit $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ with center the origin. The function $\hat{\nu}^{\varkappa}(\,\cdot\,)$ (respectively, the function $\check{\nu}^{\varkappa}(\,\cdot\,)$) with $\varkappa = 0,-,+$ is referred [3], [4] to as the Sergeev upper (respectively, lower) frequency of zeros, signs, and roots of the equation (1), respectively. The image $\hat{\nu}^{\varkappa}(S_*(a))$ (respectively, the image $\check{\nu}^{\varkappa}(S_*(a))$) of the function $\hat{\nu}^{\varkappa}(\,\cdot\,)$ (respectively, the function $\check{\nu}^{\varkappa}(\,\cdot\,)$) is referred to as the upper (respectively, lower) frequency spectra of zeros if $\varkappa = 0$, signs if $\varkappa = -$, and roots if $\varkappa = +$.

The descriptions of the Baire classes and the spectra of the Sergeev upper frequency of zeros, signs, and roots of the equation (1) were provided in [2]. In this paper we present results on the Baire classes and structure of the spectra of the Sergeev lower frequency of zeros, signs, and roots of the equation (1).

To formulate our results let us briefly give some necessary notations and definitions. Let $f(\,\cdot\,)$ be a real- or $\overline{\mathbb{R}}$-valued function defined on some set $\mathcal{M}$. For each number $r \in \mathbb{R}$ and for a function $f(\,\cdot\,)$, the Lebesgue sets $[f > r]$ and $[f \geqslant r]$ are defined as the sets $[f > r] = \{t \in \mathcal{M} \colon f(t) > r\}$ and $[f \geqslant r] = \{t \in \mathcal{M} \colon f(t) \geqslant r\}$. The sets $[f < r]$ and $[f \leqslant r]$ have a similar meaning (the complements of the corresponding Lebesgue sets in $\mathcal{M}$), and $[f = r]$ is a level set of the function $f(\,\cdot\,)$. As usual, here we assume that $-\infty < r < +\infty$ for any $r \in \mathbb{R}$.

If $\mathcal{M}$ is a topological space, then its five first Borel classes of sets are known to be defined as follows [5, p. 192], [1, p. 108]. The zero class consists of closed and open sets (their classes are denoted by $F$ and $G$, respectively). The first class consists of sets of the type $G_\delta$ and the type $F_\sigma$ ($G_\delta$-sets and $F_\sigma$-sets) those are sets, which can be represented as countable intersections of open sets and countable unions of closed sets, respectively. The second class consists of sets of the type $F_{\sigma\delta}$ and the type $G_{\delta\sigma}$ ($F_{\sigma\delta}$-sets and $G_{\delta\sigma}$-sets) those are sets, which can be represented as countable intersections of $F_\sigma$-sets and countable unions of $G_\delta$-sets, respectively. Analogically, one can define sets of the type $G_{\delta\sigma\delta}$ and the type $F_{\sigma\delta\sigma}$, which form the third Borel class, and sets of the type $F_{\sigma\delta\sigma\delta}$ and the type $G_{\delta\sigma\delta\sigma}$ of the fourth Borel class.

Let $M$ and $N$ be some systems of subsets in $\mathcal{M}$. We say [5, pp. 223, 224] that a function $f(\,\cdot\,) \colon \mathcal{M} \to \mathbb{R}$ or $f(\,\cdot\,) \colon \mathcal{M} \to \overline{\mathbb{R}}$ belongs to the class $(M, {}^*)$, or $f(\,\cdot\,)$ is a function of the class $(M, {}^*)$ if its Lebesgue set satisfies the condition $[f > r] \in M$ for any $r \in \mathbb{R}$. If $[f \geqslant r] \in N$ for any $r \in \mathbb{R}$, then we say that the function $f(\,\cdot\,)$ belongs to the class $({}^*, N)$, or $f(\,\cdot\,)$ is a function of the class $({}^*, N)$. If a function $f(\,\cdot\,)$ belongs to each of the classes $(M, {}^*)$ and $({}^*, N)$, then we say that it belongs to the class $(M, N)$, or it is a function of the class $(M, N)$. We say ( [5, pp. 248, 249]; for $\overline{\mathbb{R}}$-valued functions see [6, p. 383]) that the function $f(\,\cdot\,) \colon \mathcal{M} \to \mathbb{R}$ or $f(\,\cdot\,) : \mathcal{M} \to \overline{\mathbb{R}}$ belongs to the first Baire class $\mathcal{B}_1$ if $f(\,\cdot\,) \in (F_\sigma, G_\delta)$, to the second Baire class $\mathcal{B}_2$ if $f(\,\cdot\,) \in (G_{\delta\sigma}, F_{\sigma\delta})$, and to the third Baire class $\mathcal{B}_3$ if $f(\,\cdot\,) \in (F_{\sigma\delta\sigma}, G_{\delta\sigma\delta})$.

A set $\mathcal{A} \in \mathbb{R}$ is called a Suslin set [5, p. 213], [6, p. 489] of the number line $\mathbb{R}$ if it is a continuous image of irrational numbers $\mathbb{I}$ with the subspace topology. The class of Suslin sets contains the class of Borel sets as a proper subclass, and at the same time it is a proper subclass of the class of Lebesgue measurable sets. A set $\mathcal{A} \in \overline{\mathbb{R}}$ is called a Suslin set of the extended real number line if it can be represented as an union of a Suslin set of $\mathbb{R}$ and some subset (including the empty subset) of two-element set $\{-\infty, +\infty\}$.

**Theorem.** *For any equation $a \in \widetilde{\mathcal{E}}^n$ its lower Sergeev frequency of zeros and signs belong to the class $(G_{\delta\sigma}, {}^*)$, and the lower frequency of roots belongs to the class $(F_\sigma, {}^*)$.*

It is quite interesting to compare this statement with the descriptions of the Baire classes of the Sergeev upper frequency of zeros, signs, and roots of the equation (1). Let us recall that for any equation $a \in \widetilde{\mathcal{E}}^n$ its upper Sergeev frequency of zeros and roots belong [3] to the class $({}^*, F_{\sigma\delta})$, and the lower frequency of signs belong to the class $({}^*, G_\delta, )$.

Since the image of any Baire function is [5, p. 255] a Suslin set, from the theorem it follows

**Corollary.** *For any equation $a \in \widetilde{\mathcal{E}}^n$ the lower frequency spectra $\check{\nu}^0(S_*(a))$, $\check{\nu}^-(S_*(a))$, and $\check{\nu}^+(S_*(a))$ of zeros, signs, and roots are Suslin sets of the nonnegative semi-axis $\overline{\mathbb{R}}_+$.*

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