Exact Extreme Bounds of Mobility of the Lower and the Upper Bohl Exponents of the Linear Differential System Under Small Perturbations of its Coefficient Matrix

E. A. Barabanov

Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: `bar@im.bas-net.by`

A. V. Konyukh

Belarus State Economic University, Minsk, Belarus
E-mail: `al3128@gmail.com`

Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geqslant 0, \tag{1}$$

of dimension $n \geqslant 2$ with uniformly bounded $(\sup\{\|A(t)\| : \ t \geqslant 0\} < +\infty)$ and piecewise continuous on the semi axle $t \geqslant 0$ coefficient matrix. We denote by $\mathcal{X}(A)$ the set of all nonzero solutions to the system (1), and by $X_A(\,\cdot\,,\,\cdot\,)$ – its Cauchy matrix. Let $\mathcal{M}_n$ be the metric space of the systems (1) with the metric of uniform convergence of their coefficients on the semi axle. The lower $\underline{\beta}[x]$ and the upper $\overline{\beta}[x]$ Bohl exponents of a solution $x(\,\cdot\,) \in \mathcal{X}(A)$ are defined, respectively, by the formulas [3, pp. 171, 172], [5]

$$\underline{\beta}[x] = \varliminf_{t-\tau\to+\infty} \frac{1}{t-\tau}\, \ln \frac{\|x(t)\|}{\|x(\tau)\|} \ \text{ and } \ \overline{\beta}[x] = \varlimsup_{t-\tau\to+\infty} \frac{1}{t-\tau}\, \ln \frac{\|x(t)\|}{\|x(\tau)\|}\,,$$

and the quantities

$$\omega_0(A) = \varliminf_{t-\tau\to+\infty} \frac{1}{t-\tau} \ln \|X_A^{-1}(t,\tau)\|^{-1} \ \text{ and } \ \Omega^0(A) = \varlimsup_{t-\tau\to+\infty} \frac{1}{t-\tau} \ln \|X_A(t,\tau)\| \tag{2}$$

are called, respectively, the lower and the upper general exponents (they are also known as singular exponents) of the system (1) [3, p. 172].

The following obvious inequalities can't be in general case replaced by equalities [1]:

$$\omega_0(A) \leqslant \inf_{x\in\mathcal{X}(A)} \underline{\beta}[x] \ \text{ and } \ \sup_{x\in\mathcal{X}(A)} \overline{\beta}[x] \leqslant \Omega^0(A);$$

in particular, it is possible, that the exponents $\omega_0(A)$ and $\Omega^0(A)$ can not be implemented on any solution of the system (1).

R. E. Vinograd proved [5] the following equalities

$$\omega_0(A) = \lim_{\varepsilon\to+0} \inf_{\|Q\|\leqslant\varepsilon} \inf_{x\in\mathcal{X}(A+Q)} \underline{\beta}[x] \quad \text{and} \quad \Omega^0(A) = \lim_{\varepsilon\to+0} \sup_{\|Q\|\leqslant\varepsilon} \sup_{x\in\mathcal{X}(A+Q)} \overline{\beta}[x], \tag{3}$$

i.e., in other words, the lower (the upper) general exponent of the system (1) is the exact lower (upper) bound of the lower (the upper) Bohl exponents of the solutions $x(\,\cdot\,) \in \mathcal{X}(A)$ under arbitrary small perturbations of coefficient matrix of the system (1).

From the geometric point of view the lower $\omega_0(A)$ and the upper $\Omega^0(A)$ general exponents of the system (1) are asymptotically accurate when $t-\tau\to+\infty$, respectively, lower bound of the minor semi axis and upper bound of the major semi axis on a logarithmic scale of family of ellipsoids $E_{t,\tau}\overset{\rm def}{=}\{\xi\in\mathbb{R}^n:\ \|X_A^{-1}(t,\tau)\xi\|=1\}$ (spectral matrix norm), which are generated by linear mappings $X_A(t,\tau)$, $t\geqslant\tau\geqslant 0$. From this point of view it seems natural to consider along with the quantities (2) the quantities

$$\omega^0(A)=\overline{\lim_{t-\tau\to+\infty}}\,\frac{1}{t-\tau}\,\ln\|X_A^{-1}(t,\tau)\|^{-1}\ \text{ and }\ \Omega_0(A)=\underline{\lim_{t-\tau\to+\infty}}\,\frac{1}{t-\tau}\,\ln\|X_A(t,\tau)\|,\tag{4}$$

which give asymptotically accurate when $t-\tau\to+\infty$, respectively, upper bound of the minor semi axis and lower bound of the major semi axis on a logarithmic scale of family of ellipsoids $E_{t,\tau}$, and find out whether the values (4) are connected by equalities similar to (3) with the Bohl exponents of solutions to the pertubed systems.

The introduced exponents $\omega^0(A)$ and $\Omega_0(A)$ are called, respectively, the junior upper and the senior lower Bohl exponents of the system (1) (according to this terminology the exponents $\omega_0(A)$ and $\Omega^0(A)$ are called the junior lower and the senior upper Bohl exponents of the system (1)). The quantities (2) and (4) complement each other and give an asymptotically accurate two-sided estimates of variation of the norms $\|X_A(t,\tau)\|$ and $\|X_A^{-1}(t,\tau)\|$ when $t-\tau\to+\infty$. The exponents (4) were introduced in the review article by the authors [2], the motivation of their consideration was described above. In the paper [2] the authors, being based only on the formulas (3) and the mentioned above analogy of the quantities (2) and (4), gave without proof, due to the style of the mentioned paper, the following, similar to (3), formulas, which connect the exponents (4) of the system (1) and the Bohl exponents of perturbed systems

$$\omega^0(A)=\lim_{\varepsilon\to+0}\inf_{\|Q\|\leqslant\varepsilon}\inf_{x\in\mathcal{X}(A+Q)}\overline{\beta}(x)\ \text{ and }\ \Omega_0(A)=\lim_{\varepsilon\to+0}\sup_{\|Q\|\leqslant\varepsilon}\sup_{x\in\mathcal{X}(A+Q)}\underline{\beta}[x]\tag{5}$$

considering that the proof of these equalities is completely analogous to the proof of the equalities (3) from paper [5], and even attributing it to the paper [5]. It appears that in general case the equalities (5) don't take place, as the following theorem shows.

Theorem 1. *The inequalities*

$$\omega^0(A)\geqslant\lim_{\varepsilon\to+0}\inf_{\|Q\|\leqslant\varepsilon}\inf_{x\in\mathcal{X}(A+Q)}\overline{\beta}[x]\ \textit{ and }\ \Omega_0(A)\leqslant\lim_{\varepsilon\to+0}\sup_{\|Q\|\leqslant\varepsilon}\sup_{x\in\mathcal{X}(A+Q)}\underline{\beta}[x]\tag{6}$$

are valid, and for every natural $n\geqslant 2$ there exist such systems (1) *for which each of these inequalities is strict.*

Let us denote by $\omega_*^0(A)$ and $\Omega_0^*(A)$ the right sides of the inequalities (6) respectively, in other words the exponent $\omega_*^0(A)$ is the exact lower bound of the upper Bohl exponents, and the exponent $\Omega_0^*(A)$ is the exact upper bound of the lower Bohl exponents of the solutions $x(\,\cdot\,)\in\mathcal{X}(A)$ under arbitrary small perturbations of coefficient matrix of the system (1). The exact expressions for the quantities $\omega_*^0(A)$ and $\Omega_0^*(A)$ using the Cauchy matrix of the system (1) are given in the following theorem.

Theorem 2. *The equalities*

$$\omega_*^0(A)=\lim_{T\to+\infty}\overline{\lim_{k-m\to+\infty}}\,\frac{1}{(k-m)T}\sum_{i=m+1}^{k}\ln\|X_A^{-1}(iT,(i-1)T)\|^{-1},$$

$$\Omega_0^*(A)=\lim_{T\to+\infty}\underline{\lim_{k-m\to+\infty}}\,\frac{1}{(k-m)T}\sum_{i=m+1}^{k}\ln\|X_A(iT,(i-1)T)\|,$$

where $k,m\in\mathbb{N}$, are valid.

The fact that the right sides of these equalities are correctly defined (i.e. that the outer limits in the right sides of these equalities exist), is established in the proof of Theorem 2.

The mentioned above theorem by R. E. Vinograd [5] (see the relations (2) and (3)) and Theorem 2 give the formulas for calculating, using the Cauchy matrix of the system (1), of the exact extreme bounds of variation (mobility) of the upper and the lower Bohl exponents of the solutions under small perturbations of its coefficient matrix. Consider how these exact bounds $\Omega^0(A)$, $\omega^0_*(A)$ and $\Omega^*_0(A)$, $\omega_0(A)$, as well as the quantities $\Omega_0(A)$ and $\omega^0(A)$, can vary themselves under small perturbations of the coefficient matrix of the system (1). Let us recall that a real-valued function, defined on a metric space $\mathcal{M}_n$, is called upwards stable (downwards stable), if it is upper (respectively, lower) semicontinuous function on this space.

The exponent $\Omega^0(\,\cdot\,)$ is upwards stable, and the exponent $\omega_0(\,\cdot\,)$ is downwards stable [3, p. 180], but they are both unstable in the opposite directions, if $n \geqslant 2$ [4]. The exponents $\Omega^*_0(A)$ and $\omega^0_*(A)$ possess the same properties, as the following theorems show, but neither $\Omega_0(A)$ nor $\omega^0(A)$ do.

Theorem 3. *The exponent $\Omega^*_0(\,\cdot\,)$ is upwards stable, and the exponent $\omega^0_*(\,\cdot\,)$ is downwards stable.*

Theorem 4. *If $n \geqslant 2$, the exponent $\Omega^*_0(\,\cdot\,)$ is downwards unstable, and the exponent $\omega^0_*(\,\cdot\,)$ is upwards unstable, i.e. for $n \geqslant 2$ there exist such systems $A \in \mathcal{M}_n$, for which the inequalities*

$$\lim_{\varepsilon\to+0}\inf_{\|Q\|\leqslant\varepsilon}\Omega^*_0(A+Q) < \Omega^*_0(A) \quad \textit{and} \quad \lim_{\varepsilon\to+0}\sup_{\|Q\|\leqslant\varepsilon}\omega^0_*(A+Q) > \omega^0_*(A)$$

hold, respectively.

Theorem 5. *Each of the exponents $\Omega_0(A)$ and $\omega^0(A)$ is neither upwards, nor downwards stable under small perturbations of the coefficient matrix.*

References

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