On the Cauchy Problem for Linear Systems of Generalized Ordinary Differential Equations with Singularities

Malkhaz Ashordia

A. Razmadze Mathematical Institute of I. Javakhishvili Tbiisi State University, Tbilisi, Georgia; Sokhumi State University, Tbilisi, Georgia

E-mail: ashord@rmi.ge

Let $I \subset \mathbb{R}$ be an interval non-degenerate in the point, $t_0 \in \mathbb{R}$ and

$$I_{t_0} = I \setminus \{t_0\}.$$

Consider the linear system of generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \quad \text{for} \quad t \in I_{t_0}, \tag{1}$$

where $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$, $f = (f_k)_{k=1}^n \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^n)$.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_{t_0} \to \mathbb{R}^{n\times n}$ be a diagonal matrix-function with continuous diagonal elements $h_k : I_{t_0} \to]0, +\infty[$ $(k = 1, \dots, n)$.

We consider the problem of finding a solution $x \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^n)$ of the system (1), satisfying the condition

$$\lim_{t\to t_0-} (H^{-1}(t)\, x(t)) = 0 \;\; \text{and} \;\; \lim_{t\to t_0+} (H^{-1}(t)\, x(t)) = 0. \tag{2}$$

The analogous problem for systems of ordinary differential equations with singularities

$$\frac{dx}{dt} = P(t)\, x + q(t) \quad \text{for} \quad t \in I, \tag{3}$$

where $P \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$, $q \in L_{loc}(I_{t_0}, \mathbb{R}^n)$, are investigated in [5–7].

The singularity of system (3) is considered in the sense that the matrix P and vector q functions, in general, are not integrable at the point t_0. In general, the solution of the problem (3), (2) is not continuous at the point t_0 and, therefore, it is not a solution in the classical sense. But its restriction to every interval from I_{t_0} is a solution of the system (3). In connection with this we give the example from [7].

Let $\alpha > 0$ and $\varepsilon \in]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha\, x}{t} + \varepsilon |t|^{\varepsilon - 1\alpha}, \quad \lim_{t\to 0} \alpha(t^\alpha x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not a solution of the equation on the set $I = \mathbb{R}$, but its restrictions to $]-\infty, 0[$ and $]0, +\infty[$ are solutions of that one.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, [1–4, 8, 9].

We give sufficient conditions for the unique solvability of the problem (1), (2). The analogous results for the Cauchy problem for systems of ordinary differential equations with singularities belong to I. Kiguradze ([6, 7]).

In the paper, the use will be made of the following notation and definitions.

$\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$, $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$.

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix.

If $X=(x_{ij})_{i,j=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_{\mp}=\frac{1}{2}\,(|X|\mp X)$.

$\mathbb{R}_+^{n\times m}=\big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\big\}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t)=(v(x_{ij})(t))_{i,j=1}^{n,m}$ for $t\in I$, where $v(x_{ij})(a)=0$, $v(x_{ij})(t)\equiv\bigvee\limits_a^t(x_{ij})$, and $a\in\mathbb{R}$ is some fixed point; $[X(t)]_+^v\equiv\frac{1}{2}(V(X)(t)+X(t))$, $[X(t)]_-^v\equiv\frac{1}{2}(V(X)(t)-X(t))$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X:[a,b]\to\mathbb{R}^{n\times m}$ at the point t $(X(a-)=X(a),X(b+)=X(b))$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<\infty$).

$\mathrm{BV}_{loc}(J;D)$, where $J\subset\mathbb{R}$ is an interval and $D\subset\mathbb{R}^{n\times m}$, is the set of all $X:J\to D$ for which the restriction to $[a,b]$ belong to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from J;

$\mathrm{BV}_{loc}(I_{t_0};D)$ is the set of all $X:I\to D$ for which the restriction to $[a,b]$ belong to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from I_{t_0};

s_1, s_2 and $s_c:\mathrm{BV}_{loc}(J;\mathbb{R})\to\mathrm{BV}_{loc}(J;\mathbb{R})$ are the operators defined by

$$s_1(x)(a)=s_2(x)(a)=0,\ \ s_0(x)=x(a);$$
$$s_1(x)(t)=s_1(x)(s)+\sum_{s<\tau\leq t}d_1x(\tau),\ \ s_2(x)(t)=s_2(x)(s)+\sum_{s\leq\tau<t}d_2x(\tau),$$
$$s_0(x)(t)=s_0(x)(s)+x(t)-x(s)-\sum_{j=1}^{2}(s_j(x)(t)-s_j(x)(s))\ \text{ for }\ s<t,$$

where $a\in J$ is an arbitrarily fixed point.

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\leq s<t\leq b$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_0(g)(\tau)+\sum_{s<\tau\leq t}x(\tau)d_1g(\tau)+\sum_{s\leq\tau<t}x(\tau)d_2g(\tau),$$

where $\int\limits_{]s,t[}x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$. So $\int\limits_s^t x(\tau)\,dg(\tau)$ is the Kurzweil integral [8,9]; We put

$$\int_{s\mp}^t x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int_{s\mp\delta}^t x(\tau)\,dg(\tau).$$

If $X \in \mathrm{BV}_{loc}(J;\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^j d_j X(t)) \neq 0$ for $t\in I$ $(j=1,2)$, and $Y \in \mathrm{BV}_{loc}(J;\mathbb{R}^{n\times m})$, then

$$\begin{aligned}\mathcal{A}(X,Y)(a) &= O_{n\times m},\\ \mathcal{A}(X,Y)(t)-\mathcal{A}(X,Y)(s) &= Y(t)-Y(s)+\sum_{s<\tau\leq t} d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\ &\qquad -\sum_{s\leq\tau<t} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ for } s<t.\end{aligned}$$

A vector-function $x: I_{t_0}\to\mathbb{R}^n$ is said to be a solution of the system (1) if $x\in \mathrm{BV}([a,b],\mathbb{R}^n)$ for every closed interval $[a,b]$ from I_{t_0} and

$$x(t)=x(s)+\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for } a\leq s<t\leq b.$$

We assume that

$$\det(I_n+(-1)^j d_jA(t))\neq 0\ \text{ for } t\in I_{t_0}\ (j=1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $A\in \mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ and $f\in \mathrm{BV}_{loc}(I,\mathbb{R}^n)$.

Let $A_0\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$. Then a matrix-function $C_0: I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the generalized differential system

$$dx=dA_0(t)\cdot x, \tag{4}$$

if, for every interval and $J\subset I$ and $\tau\in J$, the restriction of the matrix-function $C_0(\,\cdot\,,\tau): I_{t_0}\to\mathbb{R}^{n\times n}$ to J is the fundamental matrix of the system (4), satisfying the condition $C_0(\tau,\tau)=I_n$. Therefore, C_0 is the Cauchy matrix of (4) if and only if the restriction of C_0 to the every interval $J\times J$ is the Cauchy matrix of the system in the sense of definition given in [9].

We assume $I_{t_0}^-=]-\infty,t_0[\cap I$, $I_{t_0}^+=]t_0,+\infty[\cap I$ and $I_{t_0}^-(\delta)=[t_0-\delta,t_0[\cap I_{t_0}$, $I_{t_0}^+(\delta)=]t_0,t_0+\delta]\cap I_{t_0}$, $I_{t_0}(\delta)=I_{t_0}^-(\delta)\cup I_{t_0}^+(\delta)$ for every $\delta>0$.

Theorem 1. *Let there exist a matrix-function $A_0\in \mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and constant matrices $B_0,\ B\in\mathbb{R}_+^{n\times n}$ such that*

$$\begin{gathered}\det\big(I_n+(-1)^j d_jA_0(t)\big)\neq 0\ \textit{ for } t\in I_{t_0}\ (j=1,2),\\ r(B)<1,\end{gathered} \tag{5}$$

and the estimates

$$|C_0(t,\tau)|\leq H(t)\,B_0\,H^{-1}(\tau)\ \textit{ for } t\in I_{t_0}(\delta),\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\leq|t-t_0|;$$

$$\bigg|\int_{t_0\mp}^t |C_0(t,\tau)|\,dV(\mathcal{A}(A_0,A-A_0)(\tau))\cdot H(\tau)\bigg|\leq H(t)B\ \textit{ for } t\in I_{t_0}^-(\delta)\ \textit{and}\ t\in I_{t_0}^+(\delta),\ \textit{respectively,}$$

hold for some $\delta>0$, where C_0 is the Cauchy matrix of the system (4)*. Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\bigg\|\int_{t_0\mp}^t H^{-1}(\tau)\,C_0(t,\tau)\,d\mathcal{A}(A_0,f)(\tau)\bigg\|=0.$$

Then the problem (1), (2) *has a unique solution.*

Theorem 2. *Let there exist a matrix* $B = (b_{ik})_{i,k=1}^{n} \in \mathbb{R}_+^{n\times n}$ *such that the condition* (5) *and*

$$\big[(-1)^j d_j a_{ii}(t)\big]_- < 1 \ \ \text{for} \ \ (-1)^j(t-t_0) > 0 \ \ (j=1,2;\ i=1,\dots,n),$$

hold, and the estimates

$$|c_i(t,\tau)| \le b_0\,\frac{h_i(t)}{h_i(\tau)} \ \ \text{for} \ \ t \in I_{t_0}(\delta), \ \ (t-t_0)(\tau-t_0) > 0, \ \ |\tau - t_0| \le |t-t_0| \ \ (i=1,\dots,n),$$

$$\bigg|\int\limits_{t_0\mp}^{t} c_i(t,\tau)h_i(\tau)\,d\big[a_{ii}(\tau)\operatorname{sgn}(\tau-t_0)\big]_+^v\bigg| \le b_{ii}(t)h_i(t)$$

$$\text{for } t \in I_{t_0}^-(\delta) \ \text{ and } t \in I_{t_0}^+(\delta), \text{ respectively } (i=1,\dots,n);$$

$$\bigg|\int\limits_{t_0\mp}^{t} c_i(t,\tau)h_k(\tau)\,dV(\mathcal{A}(a_{0ii},a_{ik}))(\tau)\bigg| \le b_{ik}(t)h_i(t)$$

$$\text{for } t \in I_{t_0}^-(\delta) \ \text{ and } t \in I_{t_0}^+(\delta), \text{ respectively } (i\neq k;\ i,k=1,\dots,n)$$

hold for some $b_0 > 0$ *and* $\delta > 0$. *Let, moreover, respectively,*

$$\lim_{t\to t_0\mp}\int\limits_{t_0\mp}^{t} \frac{c_i(t,\tau)}{h_i(t)}\,dV(\mathcal{A}(a_{0ii},f_i))(\tau) = 0 \ \ (i=1,\dots,n),$$

where $a_{0ii}(t) \equiv -[a_{ii}(t)\operatorname{sgn}(t-t_0)]_-^v\operatorname{sgn}(t-t_0)$ $(i=1,\dots,n)$, *and* c_i *is the Cauchy function of the equation* $dx = x\,da_{0ii}(t)$ *for* $i \in \{1,\dots,n\}$. *Then the problem* (1), (2) *has a unique solution.*

Remark. The Cauchy functions $c_i(t,\tau)$ $(i=1,\dots,n)$, mentioned in the theorem, for $t,\tau \in I_{t_0}^-$ and $t,\tau \in I_{t_0}^+$, have the form

$$c_i(t,\tau) = \begin{cases} \exp\big(s_0(a_{0ii})(t) - s_0(a_{0ii})(\tau)\big)\prod\limits_{\tau<s\le t}(1-d_1a_{0ii}(s))^{-1}\prod\limits_{\tau\le s<t}(1+d_2a_{0ii}(s)) & \text{for } \ t>\tau, \\ \exp\big(s_0(a_{0ii})(t) - s_0(a_{0ii})(\tau)\big)\prod\limits_{t<s\le \tau}(1-d_1a_{0ii}(s))\prod\limits_{t\le s<\tau}(1+d_2a_{0ii}(s))^{-1} & \text{for } \ t<\tau. \end{cases}$$

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