# On the Cauchy Problem for Linear Systems of Impulsive Equations with Singularities

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Let $I \subset \mathbb{R}$ be an interval non-degenerate in the point, $t_0 \in \mathbb{R}$ and

$$I_{t_0} = I \setminus \{t_0\}.$$

Consider the linear system of impulsive equations with fixed and finite points of impulses actions

$$\frac{dx}{dt} = P(t)\,x + q(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus \{\tau_l\}_{l=1}^{\infty}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = G_l\,x(\tau_l) + g_l \ \ (l = 1, 2, \dots), \tag{2}$$

where $P \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$, $q \in L_{loc}(I_{t_0}, \mathbb{R}^n)$, $G_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $g_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$, $\tau_l \in I_{t_0}$ $(l = 1, 2, \dots)$, $\tau_i \neq \tau_j$ if $i \neq j$ and $\lim\limits_{l\to\infty} \tau_l = t_0$.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_{t_0} \to \mathbb{R}^{n\times n}$ be a diagonal matrix-functions with continuous diagonal elements $h_k : I_{t_0} \to ]0, +\infty[$ $(k = 1, \dots, n)$.

We consider the problem of finding a solution $x : I_{t_0} \to \mathbb{R}^n$ of the system (1), (2), satisfying the condition

$$\lim_{t\to t_0} \big(H^{-1}(t)x(t)\big) = 0. \tag{3}$$

The analogous problem for the systems (1) of ordinary differential equations with singularities are investigated in [2–4].

The singularity of the system (1) is considered in the sense that the matrix $P$ and vector $q$ functions, in general, are not integrable at the point $t_0$. In general, the solution of the problem (1), (3) is not continuous at the point $t_0$ and, therefore, it is not a solution in the classical sense. But its restriction to every interval from $I_{t_0}$ is a solution of the system (1). In connection with this we give the example from [4].

Let $\alpha > 0$ and $\varepsilon \in ]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha\,x}{t} + \varepsilon |t|^{\varepsilon - 1\alpha},$$
$$\lim_{t\to 0} (t^{\alpha} x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not a solution of the equation on the set $I = \mathbb{R}$, but its restrictions to $] - \infty, 0[$ and $]0, +\infty[$ are solutions of that equation.

We give sufficient conditions for the unique solvability of the problem $(1),(2);(3)$. The analogous results belong to I. Kiguradze [3, 4] for the Cauchy problem for systems of ordinary differential equations with singularities.

Some boundary value problems for linear impulsive systems with singularities are investigated in [1] (see, also the references herein).

In the paper, the use will be made of the following notation and definitions.

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R} = ]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$, $[a,b]$ and $]a,b[$ $(a,b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times m}$ (or $O$) is the zero $n\times m$ matrix.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column $n$-vectors $x = (x_i)_{i=1}^n$.

If $X\in\mathbb{R}^{n\times n}$, then $X^{-1}$, $\det X$ and $r(X)$ are, respectively, the matrix inverse to $X$, the determinant of $X$ and the spectral radius of $X$; $I_n$ is the identity $n\times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X : [a,b]\to\mathbb{R}^{n\times m}$ is a matrix-function, then $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b]\to\mathbb{R}^{n\times m}$ at the point $t$ $(X(a-) = X(a)$, $X(b+) = X(b))$.

$\widetilde{C}([a,b],D)$, where $D\subset\mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b]\to D$.

$\widetilde{C}_{loc}(I_{t_0}\setminus\{\tau_l\}_{l=1}^{\infty},D)$ is the set of all matrix-functions $X : I_{t_0}\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I_{t_0}\setminus\{\tau_l\}_{l=1}^{\infty}$ belong to $\widetilde{C}([a,b],D)$.

$L([a,b];D)$ is the set of all integrable matrix-functions $X : [a,b]\to D$.

$L_{loc}(I_{t_0};D)$ is the set of all matrix-functions $X : I_{t_0}\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I_{t_0}$ belong to $L([a,b],D)$.

A vector-function $x\in\widetilde{C}_{loc}(I_{t_0}\setminus\{\tau_l\}_{l=1}^{\infty},\mathbb{R}^n)$ is said to be a solution of the system $(1),(2)$ if

$$x'(t) = P(t)\,x(t) + q(t) \ \text{ for a.a. } \ t\in I_{t_0}\setminus\{\tau_l\}_{l=1}^{\infty}$$

and there exist one-sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l=1,2,\dots)$ such that the equalities (2) hold.

We assume that

$$\det(I_n + G_l)\neq 0 \ \ (l=1,2,\dots).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $P\in L_{loc}(I,\mathbb{R}^{n\times n})$ and $q\in L_{loc}(I,\mathbb{R}^n)$.

Let $P_0\in L_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and $G_{0l}\in\mathbb{R}^{n\times n}$ $(l=1,2,\dots)$. Then a matrix-function $C_0 : I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous impulsive system

$$\frac{dx}{dt} = P_0(t)x, \tag{4}$$

$$x(\tau_l+) - x(\tau_l-) = G_{0l}x(\tau_l) \ \ (l=1,2,\dots), \tag{5}$$

if for every interval $J\subset I_{t_0}$ and $\tau\in J$ the restriction of the matrix-function $C_0(\,\cdot\,,\tau):I_{t_0}\to\mathbb{R}^{n\times n}$ to $J$ is the fundamental matrix of the system (4), (5) satisfying the condition $C_0(\tau,\tau)=I_n$. Therefore, $C_0$ is the Cauchy matrix of (4), (5) if and only if the restriction of $C_0$ on $J\times J$, for every interval $J\subset I_{t_0}$, is the Cauchy matrix of the system in the sense of definition given in [5].

We assume $I_{t_0}(\delta)=[t_0-\delta,t_0+\delta]\cap I_{t_0}$ for every $\delta>0$.

**Theorem.** *Let there exist a matrix-function $P_0\in L_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and constant matrices $G_l\in\mathbb{R}^{n\times n}$ $(l=1,2,\dots)$ and $B_0,B\in\mathbb{R}_+^{n\times n}$ such that*

$$\det(I_n+G_{0l})\neq 0\ \ (l=1,2,\dots),\quad r(B)<1,$$

*and the estimates*

$$|C_0(t,\tau)|\le H(t)B_0H^{-1}(\tau)\ \ \textit{for}\ \ t\in I_{t_0}(\delta),\quad (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\le|t-t_0|$$

*and*

$$\bigg|\int\limits_{t_0}^{t}\big|C_0(t,\tau)(P(\tau)-P_0(\tau))H(\tau)\big|\,d\tau\bigg| \\ +\bigg|\sum_{l\in\mathcal{N}_{t_0,t}}\big|C_0(t,\tau_l)G_{0l}(I_n+G_{0l})^{-1}(G_l-G_{0l})\big|\bigg|\le H(t)B\ \ \textit{for}\ \ t\in I_{t_0}(\delta)$$

*hold for some $\delta>0$, where $C_0$ is the Cauchy matrix of the system* (4), (5)*. Let, moreover,*

$$\lim_{t\to t_0}\bigg\|\int\limits_{t_0}^{t}H^{-1}(\tau)C_0(t,\tau)q(\tau)\,d\tau+\sum_{l\in\mathcal{N}_{t_0,t}}H^{-1}(\tau_l)C_0(t,\tau_l)G_{0l}(I_n+G_{0l})^{-1}g_l\bigg\|=0.$$

*Then the problem* (1), (2); (3) *has the unique solution.*

## References


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