Recent Development of Boundary Value Problems of q-Difference and Fractional q-Difference Equations and Inclusions

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1 Introduction

The subject of q-calculus, also known as quantum calculus, rests on the concept of finite difference re-scaling. The formal work on q-difference equations dates back to the first quarter of twentieth century. The applications of q-calculus in several important disciplines like combinatorics, special functions, quantum mechanics, etc. led to the recent development of the subject. q-calculus is also regarded as a subfield of time scales calculus (unified setting for studying dynamic equations on both discrete and continuous domains). In this short note, we present some recent results on boundary value problems (BVP) of q-difference and fractional q-difference equations and inclusions.

2 BVP for q-difference equations and inclusions

We begin with some preliminary concepts of q-calculus.

Definition 2.1. Let f be a function defined on a q-geometric set I, i.e., $qt \in I$ for all $t \in I$. For $0 < q < 1$, we define the q-derivative as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}\,, \ \ t \in I \setminus \{0\}, \ \ D_q f(0) = \lim_{t\to 0} D_q f(t).$$

For $t \geq 0$, we consider a set $J_t = \{tq^n : \ n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and define the definite q-integral of a function $f : J_t \to \mathbb{R}$ by

$$I_q f(t) = \int\limits_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

For $a,b\in J_t$, we have

$$\int\limits_a^b f(s)\,d_qs=I_qf(b)-I_qf(a)=(1-q)\sum_{n=0}^{\infty}q^n\big[bf(bq^n)-af(aq^n)\big].$$

Consider the boundary value problem for a second order $q-$difference equation with non-separated boundary conditions

$$D_q^2x(t)=f(t,x(t)),\ \ t\in I,\quad x(0)=\eta x(T),\quad D_qx(0)=\eta D_qx(T), \tag{2.1}$$

where $f\in C(I\times\mathbb{R},\mathbb{R})$, $I=[0,T]\cap\{q^n:\ n\in\mathbb{N}\}\cup\{0\}$, T is a fixed constant and $\eta\neq 1$ is a fixed real number. By using a variety of fixed point theorems such as Banach's contraction principle, Leray–Schauder nonlinear alternative, Schauder fixed point theorem and Krasnoselskii's fixed point theorem, several results are proved for the problem (2.1) in [5], which are listed below.

Theorem 2.2. *Let $f:I\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the condition*

$$|f(t,u)-f(t,v)|\le L|u-v|,\ \ \forall\,t\in I,\ \ u,v\in\mathbb{R},$$

where L is a Lipschitz constant. Then the boundary value problem (2.1) *has a unique solution, provided*

$$L\Big(\frac{1}{1+q}+\frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2}+\Big|\frac{\eta}{\eta-1}\Big|\Big)T^2<1.$$

Theorem 2.3. *Assume that:*

(H_1) *there exists a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a function $p\in L^1([0,T],\mathbb{R}_+)$ such that*

$$|f(t,u)|\le p(t)\psi(\|u\|)\ \ \text{for each}\ \ (t,u)\in I\times\mathbb{R};$$

(H_2) *there exists a number $M>0$ such that*

$$\|u\|\Big/\Big(T\Big(1+\frac{|\eta|(1+|1-\eta|)}{(\eta-1)^2}\Big)\psi(M)\|p\|_{L^1}\Big)>1.$$

Then the BVP (2.1) *has at least one solution.*

Theorem 2.4. *Assume that there exist constants*

$$0\le c<1\Big/\Big(\frac{1}{1+q}+\frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2}+\Big|\frac{\eta}{\eta-1}\Big|\Big)$$

and $N>0$ such that $|f(t,u)|\le\frac{c}{T^2}\,|u|+N$ for all $t\in I$, $u\in C(I)$. Then the BVP (2.1) *has at least one solution.*

Theorem 2.5. *Assume that there exists a constant M_1 such that*

$$|f(t,u)|\le M_1\Big/\Big(\frac{1}{1+q}+\frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2}+\Big|\frac{\eta}{\eta-1}\Big|\Big)T^2,\ \ \forall\,t\in I,\ \ u\in[-M_1,M_1].$$

Then the BVP (2.1) *has at least one solution.*

Theorem 2.6. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the following assumptions hold:*

(H_3) $|f(t,u) - f(t,v)| \le L|u - v|$, $\forall\, t \in I$, $u, v \in \mathbb{R}$;

(H_4) $|f(t,u)| \le \mu(t)$, $\forall\,(t,u) \in I \times \mathbb{R}$, *and* $\mu \in C(I, \mathbb{R}^+)$.

If

$$\Big(\frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2} + \Big|\frac{\eta}{\eta-1}\Big|\Big)T^2 < 1,$$

then the boundary value problem (2.1) *has at least one solution on I.*

In [4], the authors discussed the existence and nonexistence of solutions for nonlinear second order q-integro-difference equation: $D_q^2 u(t) = f(t, u(t)) + I_q g(t, u(t))$, $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ supplemented with non-separated boundary conditions given in (2.1). Similar results were proved for other classes of boundary value problems. The results for the second order q-difference equation $D_q^2 x(t) = f(t, x(t))$, $t \in I$, supplemented with non-separated boundary conditions $\alpha_1 x(0) - \beta_1 D_q x(0) = \gamma_1 x(\eta_1)$, $\alpha_2 x(1) - \beta_2 D_q x(1) = \gamma_2 x(\eta_2)$ were proved in [13], with three-point integral boundary conditions $\alpha x(\eta) + \beta D_r x(\eta) = 0$, $\int\limits_0^T x(s)\, d_p s = 0$ in [22], nonlocal and integral boundary conditions

$$x(0) = x_0 + g(x), \quad x(1) = \alpha \int\limits_\mu^\nu x(s)\, d_q s,$$

and

$$x(\xi) = g(x), \quad \alpha D_r x(\eta) + \beta \int\limits_\eta^T x(s)\, d_p s = 0$$

in [8] and [18], respectively. For results on inclusions, see [7] and [17].

Boundary value problems for nonlinear q-difference hybrid equations and inclusions were studied in [11]. In [11] the authors have investigated the problem:

$$D_q^2\Big(\frac{x(t)}{f(t, x(t))}\Big) = g(t, x(t)), \ \ t \in I_q, \ \ x(0) = 0, \ \ x(1) = 0,$$

where $f \in C(I_q \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g : C(I_q \times \mathbb{R}, \mathbb{R})$ are such that $f(t, x(t)), g(t, x(t))$ are continuous at $t = 0, 1$, $I_q = \{q^n : n \in \mathbb{N}\} \cup \{0, 1\}$, $q \in (0, 1)$ is a fixed constant. An existence result was established by using a fixed point theorem for the product of two operators under Lipschitz and Carathéodory conditions.

Agarwal *et al.* [3] discussed the existence, uniqueness and existence of extremal solutions for a nonlinear boundary value problem of q-difference equations with nonlocal q-integral boundary condition given by

$$D_q u(t) = f\big(t, u(t), u(\phi(t))\big), \quad u(0) = \lambda \int\limits_0^\eta g(s, u(s))\, d_q s + \mu, \ \ t \in I_q,$$

where $f \in C(I_q \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(I_q \times \mathbb{R}, \mathbb{R})$, $\phi \in C(I_q, I_q)$, $\eta \ge 0$, $\lambda, \mu \in \mathbb{R}$, $I_q = \{q^n : n \in \mathbb{N}\} \cup \{0, 1\}$, $q \in (0, 1)$ is a fixed constant.

The notions of q-derivative and q-integral were extended on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$, let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows:

Definition 2.7. Let $f : J_k \to \mathbb{R}$ be a continuous function and let $t \in J_k$. Then we define the q_k-derivative of the function f as

$$D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}\,, \quad t \neq t_k, \quad D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t).$$

We say that f is q_k-differentiable on J_k provided $D_{q_k} f(t)$ exists for all $t \in J_k$.

Definition 2.8. Let $f : J_k \to \mathbb{R}$ be a continuous function. Then the q_k-integral is defined by

$$\int\limits_{t_k}^{t} f(s)\, d_{q_k} s = (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + (1-q_k^n)t_k\big) \tag{2.2}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$, then the definite q_k-integral is defined by

$$\int\limits_{a}^{t} f(s)\, d_{q_k} s = (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + (1-q_k^n)t_k\big) - (1-q_k)(a-t_k) \sum_{n=0}^{\infty} q_k^n f\big(q_k^n a + (1-q_k^n)t_k\big).$$

For more details on these two new notions, the interested reader is referred to the book [15].

Agarwal *et al.* [2] obtained the positive extremal solutions by the method of successive iterations for the nonlinear impulsive q_k-difference equations:

$$\begin{gathered} D_{q_k} u(t) = f(t, u(t)), \ \ 0 < q_k < 1, \ \ t \in J', \\ u(t_k) = I_k(u(t_k)), \ \ k = 1, 2, \dots, m, \quad u(0) = \lambda u(\eta) + d, \ \ \eta \in J_r, \ \ r \in \mathbb{Z}, \end{gathered}$$

where D_{q_k} are q_k-derivatives ($k = 0, 1, 2, \dots, m$), $f \in C(J \times \mathbb{R}, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}, \mathbb{R}^+)$, $J = [0, T]$, $T > 0$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $J_r = (t_r, T]$, $0 \le \lambda < 1$, $d \ge 0$, $0 \le r \le m$ and $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ ($k = 1, 2, \cdots, m$), respectively.

3 BVP for fractional q-difference equations and inclusions

Definition 3.1. Let $\nu \ge 0$ and h be a function defined on $[0, T]$. The fractional q-integral of Riemann–Liouville type is given by $(I_q^0 h)(t) = h(t)$ and

$$(I_q^{\nu} h)(t) = \frac{1}{\Gamma_q(\nu)} \int\limits_0^t (t - qs)^{(\nu-1)} h(s)\, d_q s, \ \ \nu > 0, \ \ t \in [0, T].$$

Definition 3.2. The fractional q-derivative of Riemann–Liouville type of order $\nu \ge 0$ is defined by $(D_q^0 h)(t) = h(t)$ and $(D_q^{\nu} h)(t) = (D_q^l I_q^{l-\nu} h)(t)$, $\nu > 0$, where l is the smallest integer greater than or equal to ν.

In recent years, several existence and uniqueness results were obtained. In [1], by applying Krasnoselskii's fixed point theorem, Leray–Schauder nonlinear alternative and Banach's contraction principle, the authors studied the existence and uniqueness of solutions for the following q-anti-periodic boundary value problem of sequential q-fractional integro-differential equations:

$$\begin{gathered} {}^cD_q^{\alpha}({}^cD_q^{\gamma} + \lambda)x(t) = Af(t, x(t)) + BI_q^{\varrho} g(t, x(t)), \ \ 0 \le t \le 1, \ \ 0 < q < 1, \\ x(0) = -x(1), \quad \big(t^{(1-\gamma)} D_q x(t)\big)\Big|_{t=0} = -D_q x(1), \end{gathered}$$

where ${}^cD_q^\alpha$ and ${}^cD_q^\gamma$ denote the fractional q-derivative of the Caputo type, $0 < \alpha, \gamma \le 1$, $I_q^\rho(\,\cdot\,)$ denotes Riemann–Liouville integral with $0 < \rho < 1$, f, g are given continuous functions, $\lambda \in \mathbb{R}$ and A, B are real constants.

In [12], the existence and uniqueness results were obtained for the following boundary value problem of nonlinear fractional q-difference equations with nonlocal and sub-strip type boundary conditions:

$$\begin{gathered} {}^cD_q^\upsilon x(t) = f(t, x(t)), \ \ t \in [0,1], \ \ 1 < \upsilon \le 2, \ \ 0 < q < 1, \\ x(0) = x_0 + g(x), \quad x(\xi) = b \int\limits_\eta^1 x(s)\, d_qs, \ \ 0 < \xi < \eta < 1, \end{gathered}$$

where ${}^cD_q^\upsilon$ denotes the Caputo fractional q-derivative of order υ, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and b is a real constant. In [6], the existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions were obtained, while the positive extremal solutions for nonlinear fractional differential equations on a half-line were discussed in [23]. For further results, see [9, 10, 14, 16, 19–21].

Finally, we emphasize that the Definition 2.1 does not remain valid for impulse points t_k, $k \in \mathbb{Z}$ such that $t_k \in (qt, t)$. On the other hand, this situation does not arise for impulsive equations on q-time scales as the domains consist of isolated points covering the case of consecutive points of t and qt with $t_k \notin (qt, t)$. Due to this reason, the subject of impulsive quantum difference equations on dense domains could not be studied. In [15], the authors modified the classical quantum calculus for obtaining the first and second order impulsive quantum difference equations on a dense domain $[0, T] \subset \mathbb{R}$ through the introduction of a new q-shifting operator defined by ${}_a\Phi_q(m) = qm + (1-q)a$, $m, a \in \mathbb{R}$. For details, see [15].

References

[1] R. P. Agarwal, B. Ahmad, A. Alsaedi, and H. Al-Hutami, Existence theory for q-antiperiodic boundary value problems of sequential q-fractional integrodifferential equations. *Abstr. Appl. Anal.* **2014**, Art. ID 207547, 12 pp.

[2] R. P. Agarwal, G. Wang, B. Ahmad, L. Zhang, and A. Hobiny, Successive iteration and positive extremal solutions for nonlinear impulsive qk-difference equations. *Adv. Difference Equ.* **2015**, 2015:164, 8 pp.

[3] R. P. Agarwal, G. Wang, B. Ahmad, L. Zhang, A. Hobiny, and Sh. Monaquel, On existence of solutions for nonlinear q-difference equations with nonlocal q-integral boundary conditions. *Math. Model. Anal.* **20** (2015), no. 5, 604–618.

[4] R. P. Agarwal, G. Wang, A. Hobiny, L. Zhang, and B. Ahmad, Existence and nonexistence of solutions for nonlinear second order q-integro-difference equations with non-separated boundary conditions. *J. Nonlinear Sci. Appl.* **8** (2015), no. 6, 976–985.

[5] B. Ahmad, A. Alsaedi, and S. K. Ntouyas, A study of second-order q-difference equations with boundary conditions. *Adv. Difference Equ.* **2012**, 2012:35, 10 pp.

[6] B. Ahmad, J. J. Nieto, A. Alsaedi, and H. Al-Hutami, Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. *J. Franklin Inst.* **351** (2014), no. 5, 2890–2909.

[7] B. Ahmad and S. K. Ntouyas, Boundary value problems for q-difference inclusions. *Abstr. Appl. Anal.* **2011**, Art. ID 292860, 15 pp.

[8] B. Ahmad and S. K. Ntouyas, Boundary value problems for q-difference equations and inclusions with nonlocal and integral boundary conditions. *Math. Model. Anal.* **19** (2014), no. 5, 647–663.

[9] B. Ahmad and S. K. Ntouyas, Fractional q-difference hybrid equations and inclusions with Dirichlet boundary conditions. *Adv. Difference Equ.* **2014**, 2014:199, 14 pp.

[10] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, Existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions. *Adv. Difference Equ.* **2014**, 2014:257, 18 pp.

[11] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, Hybrid boundary value problems of q-difference equations and inclusions. *J. Comput. Anal. Appl.* **19** (2015), no. 6, 984–993.

[12] B. Ahmad, S. K. Ntouyas, A. Alsaedi, and H. Al-Hutami, Nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 26, 12 pp.

[13] B. Ahmad, S. K. Ntouyas, and I. K. Purnaras, Existence results for nonlinear q-difference equations with nonlocal boundary conditions. *Comm. Appl. Nonlinear Anal.* **19** (2012), no. 3, 59–72.

[14] B. Ahmad, S. K. Ntouyas, and I. K. Purnaras, Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. *Adv. Difference Equ.* **2012**, 2012:140, 15 pp.

[15] B. Ahmad, S. Ntouyas, and J. Tariboon, Quantum calculus. New concepts, impulsive IVPs and BVPs, inequalities. *Trends in Abstract and Applied Analysis*, 4. *World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ*, 2016.

[16] S. Asawasamrit, J. Tariboon, and S. K. Ntouyas, Existence of solutions for fractional q-integrodifference equations with nonlocal fractional q-integral conditions. *Abstr. Appl. Anal.* **2014**, Art. ID 474138, 12 pp.

[17] S. K. Ntouyas, T. Sitthiwirattham, and J. Tariboon, Existence results for q-difference inclusions with three-point boundary conditions involving different numbers of q. *Discuss. Math. Differ. Incl. Control Optim.* **34** (2014), no. 1, 41–59.

[18] S. K. Ntouyas and J. Tariboon, Nonlocal boundary value problems for q-difference equations and inclusions. *Int. J. Differ. Equ.* **2015**, Art. ID 203715, 12 pp.

[19] N. Pongarm, S. Asawasamrit, J. Tariboon, and S. K. Ntouyas, Multi-strip fractional q-integral boundary value problems for nonlinear fractional q-difference equations. *Adv. Difference Equ.* **2014**, 2014:193, 17 pp.

[20] S. Sitho, S. Laoprasittichok, S. K. Ntouyas, and J. Tariboon, Quantum difference Langevin system with nonlocal q-derivative conditions. *Int. J. Math. Math. Sci.* **2016**, Art. ID 4928314, 11 pp.

[21] S. Sitho, S. Laoprasittichok, S. K. Ntouyas, and J. Tariboon, Quantum difference Langevin equation with multi-quantum numbers q-derivative nonlocal conditions. *J. Nonlinear Sci. Appl.* **9** (2016), no. 6, 3491–3503.

[22] T. Sitthiwirattham, J. Tariboon, and S. K. Ntouyas, Boundary value problems for fractional difference equations with three-point fractional sum boundary conditions. *Adv. Difference Equ.* **2013**, 2013:296, 13 pp.

[23] L. Zhang, B. Ahmad, and G. Wang, Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half-line. *Bull. Aust. Math. Soc.* **91** (2015), no. 1, 116–128.