On Asymptotics of Solutions for Sufficiently Non-Linear Differential Equations of the Second Order

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We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_1(y)\varphi_2(y'), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\,\to\,]0,+\infty[$ is a continuous function, $\varphi_i : \Delta(Y_i^0) \to\,]0,+\infty[$ $(i=1,2)$ are twice continuously differentiable functions, where $\Delta(Y_i^0)$ is some one-sided neighborhood of the point Y_i^0, Y_i^0 equals either zero or $\pm\infty$. For these functions the following conditions are satisfied:

$$\lim_{\substack{z\to Y_1^0\\ z\in\Delta(Y_1^0)}} \frac{z\varphi_1'(z)}{\varphi_1(z)} = \lambda \quad (\lambda\in\mathbb{R}), \tag{2}$$

$$\varphi_2'(z)\neq 0 \ \text{ for } \ z\in\Delta(Y_2^0), \qquad \lim_{\substack{z\to Y_2^0\\ z\in\Delta(Y_2^0)}} \varphi_2(z) = \Phi_2^0, \ \ \Phi_2^0\in\{0,+\infty\}, \qquad \lim_{\substack{z\to Y_2^0\\ z\in\Delta(Y_2^0)}} \frac{\varphi_2''(z)\varphi_2(z)}{[\varphi_2'(z)]^2} = 1. \tag{3}$$

Conditions (2), (3) define that the function $\varphi_1(z)$ is regularly or slowly varying as $z\to Y_1^0$, and $\varphi_2(z)$ is rapidly varying as $z\to Y_2^0$ (see, E. Seneta [1]).

For power-functions and regularly varying nonlinearities φ_i $(i=1,2)$, the asymptotics of solutions for (1) are investigated in [2–10].

For equations of the type (1), in [11] the following class of monotonous solutions was introduced.

A solution y of the equation (1) is called $P_\omega(\Lambda_0)$-solution, where $-\infty\le\Lambda_0\le+\infty$, if it is defined on some interval $[t_0,\omega[\,\subset[a,\omega[$ and satisfies the conditions

$$\lim_{t\uparrow\omega} y(t) = Y_1^0, \quad \lim_{t\uparrow\omega}\varphi_2(y'(t)) = \Phi_2^0, \quad \lim_{t\uparrow\omega}\frac{\varphi_2'(y'(t))}{\varphi_2(y'(t))}\,\frac{y''(t)y(t)}{y'(t)} = \Lambda_0.$$

Earlier, in case $\Lambda_0\in\mathbb{R}\setminus\{0\}$, the asymptotics of $P_\omega(\Lambda_0)$-solutions of (1) were established in [11].

Present work is devoted to the establishment of asymptotics, as well as sufficient and necessary conditions for the existence of $P_\omega(\Lambda_0)$-solutions of (1), when $\Lambda_0=0$. In order to formulate the main result, we introduce auxiliary definitions and notations.

We determine that slowly varying function $\theta : \Delta(U^0)\to\,]0,+\infty[$, $U^0\in\{0,\pm\infty\}$ satisfies the condition S if for any continuously differentialble function $l : \Delta(U^0)\to\,]0,+\infty[$ such that

$$\lim_{\substack{z\to U^0\\ z\in\Delta(U^0)}} \frac{z\,l'(z)}{l(z)} = 0,$$

the following asymptotic representation is valid

$$\theta(zl(z)) = \theta(z)[1+o(1)] \ \text{ when } \ z\to U^0 \ \ (z\in\Delta(U^0)).$$

We introduce numbers

$$\mu_i^0 = \begin{cases} 1, & \text{if } Y_i^0=+\infty, \ \text{ or } \ Y_i^0=0 \ \text{ and } \ \Delta(Y_i^0) \ \text{ is right neighborhood of } 0,\\ -1, & \text{if } Y_i^0=-\infty, \ \text{ or } \ Y_i^0=0 \ \text{ and } \ \Delta(Y_i^0) \ \text{ is left neighborhood of } 0 \end{cases} \qquad (i=1,2).$$

These numbers define the signs of $P_\omega(0)$-solutions of (1) and their derivatives in some left neighborhood of ω.

We also define the functions

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t-\omega, & \text{if } \omega < +\infty, \end{cases} \qquad J(t) = \int\limits_A^t p(\tau)\varphi_1\big(\mu_1^0|\pi_\omega(\tau)|\big)\,d\tau,$$

where $A \in \{\omega, a\}$ and it is chosen so that the integral J tends either to zero or to ∞ as $t \uparrow \omega$.

In addition, we introduce the numbers

$$A_1^* = \begin{cases} 1, & \text{if } \omega = \infty, \\ -1, & \text{if } \omega < \infty, \end{cases} \qquad A_2^* = \begin{cases} 1, & \text{if } A = a, \\ -1, & \text{if } A = \omega. \end{cases}$$

Since the function $\varphi_1(z)$ is regularly varying of the λ-order as $z \to Y_1^0$, the following representation is valid:

$$\varphi_1(z) = |z|^\lambda \theta_1(z),$$

where the function $\theta_1(z)$ is slowly varying as $z \to Y_1^0$.

Theorem 1. *Let the function $\theta_1(z)$ satisfy the condition S. Then for the existence of $P_\omega(0)$-solutions of* (1) *it is necessary and sufficient that*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'(t)}{J(t)} = 0 \tag{4}$$

and the following conditions to be satisfied

$$\begin{gathered} A_1^* > 0 \ \text{ when } \ Y_1^0 = \pm\infty, \quad A_1^* < 0 \ \text{ when } \ Y_1^0 = 0, \\ A_2^* > 0 \ \text{ when } \ \Phi_2^0 = 0, \quad A_2^* < 0 \ \text{ when } \ \Phi_2^0 = \pm\infty, \end{gathered} \tag{5}$$

$$\mu_1^0\mu_2^0 A_1^* > 0 \quad \text{and} \quad \alpha_0\mu_2^0 A_2^* > 0. \tag{6}$$

Moreover, each solution of that kind admits the following asymptotic representation as $t \uparrow \omega$:

$$\begin{aligned} \frac{y(t)}{y'(t)} &= \pi_\omega(t)[1+o(1)], \\ \frac{1}{|y'|^\lambda \varphi_2'(y'(t))} &= -\alpha_0 J(t)[1+o(1)], \end{aligned}$$

and there exists a one-parametric family of these solutions if there is only one positive number among A_1^*, A_2^*, and a two-parametric family of these solutions if both numbers A_1^*, A_2^* are positive.

Theorem 2. *Let the functions $\theta_1(z)$, $|\psi^{-1}(z)|$ satisfy the condition S. Then each $P_\omega(0)$-solution of the differential equation* (1) *(in case of its existence) admits the following asymptotic representations as $t \uparrow \omega$:*

$$\begin{aligned} y(t) &= \mu_1^0\big|\pi_\omega(t)\psi^{-1}\big(\mu_2^0|J(t)|\big)\big|\,[1+o(1)], \\ \frac{1}{\varphi_2'(y'(t))} &= -\mu_2^0|J(t)|\,\big|\psi^{-1}\big(\mu_2^0|J(t)|\big)\big|^\lambda[1+o(1)]. \end{aligned}$$

These results could be illustrated for the equation

$$y'' = \alpha_0 p(t)|y|^\lambda |\ln|y||^\gamma e^{-\sigma|y'|^\delta}|y'|^{1-\delta}, \tag{7}$$

where $\alpha_0 \in \{1,-1\}$, $\delta, \sigma \in \mathbb{R}\setminus\{0\}$, $\lambda, \gamma \in \mathbb{R}$, $p : [a,\omega[\to]0,+\infty[$ is a continuous function.

For this equation $\varphi_1(z) = |z|^\lambda \ln^\gamma |z|$, $\varphi_2(z) = e^{-\sigma|z|^\delta}|z|^{1-\delta}$. The function $\varphi_1(z)$ is regularly varying of the λ-order as $z \to Y_2^0$. For $\delta > 0$, the function $\varphi_2(z)$ is rapidly varying as $z \to \pm\infty$, and for $\delta < 0$, the function $\varphi_2(z)$ is rapidly varying as $z \to 0$.

For (7), the $P_\omega(0)$-solution is

$$\lim_{t\uparrow\omega} \frac{yy''(t)}{|y'(t)|^{2-\delta}} = 0.$$

For the existence of $P_\omega(0)$-solution for the equation (7), it is necessary and sufficient the conditions (4)–(6) to be satisfied. Moreover, each solution of that kind admits the following asymptotic representation as $t \uparrow \omega$

$$y(t) = \mu_1^0|\pi_\omega(t)| \left|\frac{1}{\sigma} \ln|\sigma\delta J(t)|\right|^{\frac{1}{\delta}}[1 + o(1)],$$

$$y'(t) = \mu_2^0\left|\frac{1}{\sigma} \ln|\sigma\delta J(t)| + \frac{\lambda}{\sigma\delta} \ln\left|\frac{1}{\sigma} \ln|\sigma\delta J(t)|\right| + o(1)\right|^{\frac{1}{\delta}},$$

and there exists a one-parametric family of such solutions if there is only one positive number among A_1^*, A_2^*, and there exists a two-parametric family of such solutions if both numbers A_1^*, A_2^* are positive.

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