Asymptotic Behavior of Positive Solutions of Second Order Half-Linear Differential Equations with Deviating Arguments of Mixed Type

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We consider the second order half-linear differential equations

$$\big(p(t)\varphi(x'(t))\big)' \pm \sum_{i=1}^{m} q_i(t)\varphi(x(g_i(t))) \pm \sum_{j=1}^{n} r_j(t)\varphi(x(h_j(t))) = 0, \ \ t \geqq a, \tag{A$_\pm$}$$

$$(\varphi(\xi) = |\xi|^{\alpha-1}\xi = |\xi|^{\alpha}\operatorname{sgn}\xi, \ \ \alpha > 0, \ \ \xi \in \mathbb{R}, \ \ \text{Doubles sign correspondence})$$

for which the following conditions are always assumed to hold:

(a) $p, q_i, r_j : [a,\infty) \to (0,\infty)$, $a \geqq 0$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$ are continuous functions;

(b) $p(t)$ satisfies

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} < \infty,$$

and $\pi(t)$ is defined by

$$\pi(t) = \int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}}. \tag{1}$$

By a positive solution on an interval J of the differential equation (A$_\pm$) we mean a function $x : J \to (0,\infty)$ which is continuously differentiable on J together with $p(t)\varphi(x'(t))$ and satisfies (A$_\pm$) there.

Since the publication of the book ([8]) of Marić in the year 2000, the class of regularly varying functions in the sense of Karamata ([4]) is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear differential equation of the form

$$x''(t) = q(t)x(t), \ \ q(t) > 0.$$

The study of asymptotic analysis of nonoscillatory solutions of functional differential equation with deviating arguments in the framework of regularly varying functions was first attempted by Kusano and Marić ([5,6]). They established a sharp condition for the existence of a slowly varying solution of the second order functional differential equation with retarded argument of the form

$$x''(t) = q(t)x(g(t))$$

and the following functional differential equation with both retarded and advanced arguments of the form

$$x''(t) \pm q(t)x(g(t)) \pm r(t)x(h(t)) = 0,$$

where $q, r : [a,\infty) \to (0,\infty)$ are continuous functions, g, h are continuous and increasing with $g(t) < t$, $h(t) > t$ for $t \geqq a$ and $\lim\limits_{t\to\infty} g(t) = \infty$.

The Definitions and Properties of Regularly Varying Functions

Definition 1. A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be a regularly varying of index ρ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \ \text{ for any } \ \lambda > 0, \ \ \rho \in \mathbb{R}.$$

Proposition 1 (Representation Theorem)**.** *A measurable function $f : [a,\infty) \to (0,\infty)$ is regularly varying of index ρ if and only if it can be written in the form*

$$f(t) = c(t) \exp\bigg\{ \int\limits_{t_0}^{t} \frac{\delta(s)}{s}\, ds \bigg\}, \ \ t \geqq t_0,$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \ \ \textit{and} \ \ \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of SV=RV(0) is referred to as a slowly varying function. If $f \in \mathrm{RV}(\rho)$, then $f(t) = t^{\rho} L(t)$ for some $L \in$SV. Therefore, the class of slowly varying functions is of fundamental importance in the theory of regular variation. In addition to the functions tending to positive constants as $t \to \infty$, the following functions

$$\prod_{i=1}^{N} (\log_i t)^{m_i} \ \ (m_i \in \mathbb{R}), \quad \exp\Big\{ \prod_{i=1}^{N} (\log_i t)^{n_i} \Big\} \ \ (0 < n_i < 1), \quad \exp\Big\{ \frac{\log t}{\log_2 t} \Big\},$$

where $\log_1 t = \log t$ and $\log_k t = \log \log_{k-1} t$ for $k = 2, 3, \dots, N$, also belong to the set of slowly varying functions.

Proposition 2. *Let $L(t)$ be any slowly varying function. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} t^{\gamma} L(t) = \infty \ \ \textit{and} \ \ \lim_{t\to\infty} t^{-\gamma} L(t) = 0.$$

For the most complete exposition of the theory of regular variation and its applications the reader is referred to the book of Bingham, Goldie and Teugels ([1]).

The Definitions and Properties of Generalized Regularly Varying Functions

Definition 2. A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be slowly varying with respect to $1/\pi(t)$ if the function $f \circ (1/\pi(t))^{-1}$ is slowly varying in the sense of Karamata, where the function $\pi(t)$ is defined by (1) and $(1/\pi(t))^{-1}$ denotes the inverse function of $1/\pi(t)$. The totality of slowly varying functions with respect to $1/\pi(t)$ is denoted by $\mathrm{SV}_{\frac{1}{\pi}}$.

Definition 3. A measurable function $g : [a,\infty) \to (0,\infty)$ is said to be regularly varying function of index ρ with respect to $1/\pi(t)$ if the function $g \circ (1/\pi(t))^{-1}$ is regularly varying of index ρ in the sense of Karamata. The set of all regularly varying functions of index ρ with respect to $1/\pi(t)$ is denoted by $\mathrm{RV}_{\frac{1}{\pi}}(\rho)$.

Of fundamental importance is the following representation theorem for the generalized slowly and regularly varying functions, which is an immediate consequence of Proposition 1.

Proposition 3.

(i) *A function $f(t)$ is slowly varying with respect to $1/\pi(t)$ if and only if it can be expressed in the form*

$$f(t) = c(t) \exp\bigg\{ \int\limits_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}} \pi(s)}\, ds \bigg\}, \quad t \geqq t_0 \tag{2}$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \;\; \text{and} \;\; \lim_{t\to\infty} \delta(t) = 0.$$

(ii) *A function $g(t)$ is regularly varying of index ρ with respect to $1/\pi(t)$ if and only if it has the representation*

$$g(t) = c(t) \exp\bigg\{ \int\limits_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}} \pi(s)}\, ds \bigg\}, \quad t \geqq t_0 \tag{3}$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \;\; \text{and} \;\; \lim_{t\to\infty} \delta(t) = \rho.$$

If the function $c(t)$ in (2) (or (3)) is identically a constant on $[t_0,\infty)$, then the function $f(t)$ (or $g(t)$) is called normalized slowly varying (or normalized regularly varying of index ρ) with respect to $1/\pi(t)$. The totality of such functions is denoted by n-$\mathrm{SV}_{\frac{1}{\pi}}$ (or n-$\mathrm{RV}_{\frac{1}{\pi}}(\rho)$).

It is easy to see that if $g \in \mathrm{RV}_{\frac{1}{\pi}}(\rho)$ (or n-$\mathrm{RV}_{\frac{1}{\pi}}(\rho)$), then $g(t) = (1/\pi(t))^{\rho} L(t)$ for some $L \in \mathrm{SV}_{\frac{1}{\pi}}$ (or n-$\mathrm{SV}_{\frac{1}{\pi}}$).

Proposition 4. *Let $L \in SV_{\frac{1}{\pi}}$. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} \Big(\frac{1}{\pi(t)}\Big)^{\gamma} L(t) = \infty \;\; \text{and} \;\; \lim_{t\to\infty} \Big(\frac{1}{\pi(t)}\Big)^{-\gamma} L(t) = 0.$$

Main Result

In our previous paper ([3,7]) we have studied the problem of nonoscillation and asymptotic analysis of the half-linear differential equation involving nonlinear Sturm–Liouville type differential operator of the type

$$(p(t)\varphi(x'(t)))' \pm q(t)\varphi(x(t)) = 0 \tag{$\mathrm{B}_{\pm}$}$$

and the half-linear functional differential equation with deviating arguments of the mixed type

$$(\varphi(x'(t)))' \pm q(t)\varphi(x(g(t))) \pm r(t)\varphi(x(h(t))) = 0, \tag{$\mathrm{C}_{\pm}$}$$

where the functions $p(t)$, $q(t)$, $r(t)$, $g(t)$ and $h(t)$ are just as in the above equations.

Theorem A (J. Jaroš, T. Kusano and T. Tanigawa, [3])**.** *Suppose that* (1) *holds. The equation* ($\mathrm{B}_{\pm}$) *have a normalized slowly varying solution with respect to $1/\pi(t)$ and a normalized regularly varying solution of index -1 with respect to $1/\pi(t)$ if and only if*

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int\limits_{t}^{\infty} \pi(s)^{\alpha+1} q(s)\, ds = 0.$$

Theorem B (J. Manojlović and T. Tanigawa, [7]). *Suppose that*

$$\lim_{t\to\infty}\frac{g(t)}{t}=1 \ \ \textit{and} \ \ \lim_{t\to\infty}\frac{h(t)}{t}=1$$

hold. Then, the equation (C$_\pm$) *have a slowly varying solution and a regularly varying solution of index* 1 *if and only if*

$$\lim_{t\to\infty} t^{\alpha}\int_t^\infty q(s)\,ds=\lim_{t\to\infty} t^{\alpha}\int_t^\infty r(s)\,ds=0.$$

Aim of this talk is to establish a sharp condition of the existence of a normalized slowly varying solution with respect to $1/\pi(t)$ and a normalized regularly varying solution of index -1 with respect to $1/\pi(t)$ of the equation (A$_\pm$). Our main result is the following

Theorem. *Suppose that*

$$\lim_{t\to\infty}\frac{\pi(g_i(t))}{\pi(t)}=1 \ \ \textit{for} \ \ i=1,2,\dots,m$$

and

$$\lim_{t\to\infty}\frac{\pi(h_j(t))}{\pi(t)}=1 \ \ \textit{for} \ \ j=1,2,\dots,n$$

hold. The equation (A$_\pm$) *possesses a normalized slowly varying solution with respect to* $1/\pi(t)$ *and a normalized regularly varying solution of index* -1 *with respect to* $1/\pi(t)$ *if and only if*

$$\lim_{t\to\infty}\frac{1}{\pi(t)}\int_t^\infty \pi(s)^{\alpha+1}q_i(s)\,ds=0 \ \ \textit{for} \ \ i=1,2,\dots,m$$

and

$$\lim_{t\to\infty}\frac{1}{\pi(t)}\int_t^\infty \pi(s)^{\alpha+1}r_j(s)\,ds=0 \ \ \textit{for} \ \ j=1,2,\dots,n.$$

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