Leray–Schauder Degree Method in Periodic Problem for the Generalized Basset Fractional Differential Equation

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Let $T > 0$, $J = [0, T]$ and $\|x\| = \max\{|x(t)| : t \in J\}$ be the norm in $C(J)$.

In the literature [1, 2] the fractional differential equation

$$u'(t) = a\,{}^c\!D^{\alpha}u(t) + bu(t) + g(t), \ \ a \in \mathbb{R} \setminus \{0\}, \ \ \alpha \in (0, 1),$$

is called the Basset fractional differential equation.

We investigate the generalized Basset fractional differential equation

$$u'(t) = a(t)\,{}^c\!D^{\alpha}u(t) + f\big(t, u(t), {}^c\!D^{\beta}u(t)\big), \tag{1}$$

where $0 < \beta < \alpha < 1$, $a \in C(J)$, $f \in C(J \times \mathbb{R}^2)$ and ${}^c\!D$ stands for the Caputo fractional derivative. Further conditions on a and f will be given later.

Together with (1) we consider the periodic condition

$$u(0) = u(T). \tag{2}$$

We recall that the Riemann–Liouville fractional integral $I^{\gamma}x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as [1, 2]

$$I^{\gamma}x(t) = \int\limits_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\,\mathrm{d}s,$$

and the Caputo fractional derivative ${}^c\!D^{\gamma}x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : J \to \mathbb{R}$ as

$${}^c\!D^{\gamma}x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int\limits_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!}\, s^k\Big)\,\mathrm{d}s,$$

where Γ is the Euler gamma function and $n = [\gamma] + 1$, $[\gamma]$ means the integral part of γ.

In particular,

$${}^c\!D^{\gamma}x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\, (x(s) - x(0))\,\mathrm{d}s, \ \ \gamma \in (0, 1).$$

If $x \in C^1(J)$, then

$${}^c\!D^{\gamma}x(t) = \int\limits_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\, x'(s)\,\mathrm{d}s, \ \ \gamma \in (0, 1).$$

It is well known that $I^{\gamma} : C(J) \to C(J)$ for $\gamma \in (0, 1)$ and $I^{\gamma}I^{\delta}x(t) = I^{\gamma+\delta}x(t)$ for $\gamma, \delta \in (0, \infty)$, $x \in C(J)$.

We say that $u : J \to \mathbb{R}$ is *a solution of problem* (1), (2) if $u \in C^1(J)$, u satisfies (2) and (1) holds for $t \in J$.

The solvability of the periodic problem

$$u'(t) = a\,{}^c\!D^\alpha u(t) + f(t, u(t)), \quad u(0) = u(T),$$

where a is a positive constant and $\alpha \in (0,1)$, is discussed in [3].

In order to give the existence result for problem (1), (2), we introduce operators $\mathcal{H} : C(J) \times \mathbb{R} \times [0,1] \to C(J)$ and $\mathcal{S} : C(J) \times \mathbb{R} \times [0,1] \to C(J) \times \mathbb{R}$,

$$\mathcal{H}(x, \mu, \lambda)(t) = (1-\lambda)\mu + \lambda\Big(a(t)x(t) + f\big(t, \mu + I^\alpha x(t), I^{\alpha-\beta}x(t)\big)\Big)$$

and

$$\mathcal{S}(x, \mu, \lambda) = \Big(I^{1-\alpha}\mathcal{H}(x, \mu, \lambda)(t), \mu + I^\alpha x(t)\big|_{t=T}\Big).$$

The following result gives the property of $\mathcal{S}$ and the relation between solutions of the periodic problem (1), (2) and fixed points of the operator $\mathcal{S}(\,\cdot\,, \,\cdot\,, 1)$.

Lemma 1. *$\mathcal{S}$ is a completely continuous operator. If (x, μ) is a fixed point of $\mathcal{S}(\,\cdot\,, \,\cdot\,, 1)$, then*

$$u(t) = \mu + I^\alpha x(t) \ \ \text{for } t \in J$$

is a solution of the periodic problem (1), (2) *and $\mu = u(0)$.*

Lemma 2. *Let the conditions*

(H_1) *$a(t) \geq 0$ for $t \in J$, $a \not\equiv 0$;*

(H_2) *there exist positive constants c, k and l such that*

$$\begin{gathered} f(t, x, y \operatorname{sign} x) \operatorname{sign} x > 0 \ \ \text{for } t \in J, \ |x| \geq c, \ y \in [0, \infty), \\ |f(t, x, y)| \leq k\big(|x| + |y|\big) + l \ \ \text{for } t \in J, \ x, y \in \mathbb{R}, \end{gathered} \tag{3}$$

hold. Then there exists a positive constant S such that the estimate

$$\|x\| < S, \quad |\mu| < S$$

is fulfilled for all fixed points (x, μ) of the operator $\mathcal{S}(\,\cdot\,, \,\cdot\,, \lambda)$ with $\lambda \in [0,1]$.

Remark 1. Inequality (3) of (H_2) can be written in the following equivalent form

$$\begin{gathered} f(t, x, y) > 0 \ \ \text{for } t \in J, \ x \geq c, \ y \in [0, \infty), \\ f(t, x, y) < 0 \ \ \text{for } t \in J, \ x \leq -c, \ y \in (-\infty, 0]. \end{gathered}$$

Theorem 1. *Let* (H_1) *and* (H_2) *hold. Then the periodic problem* (1), (2) *has at least one solution.*

Proof. Keeping in mind Lemma 1, we need to prove that there exists a fixed point of the operator $\mathcal{S}(\,\cdot\,, \,\cdot\,, 1)$.

Let $S > 0$ be from Lemma 2 and let

$$\Omega = \Big\{(x, \mu) \in C(J) \times \mathbb{R} : \ \|x\| < S, \ |\mu| < S\Big\}.$$

Then Lemma 2 guarantees that

$$\mathcal{S}(x, \mu, \lambda) \neq (x, \mu) \ \ \text{for } (x, \mu) \in \partial\Omega \text{ and } \lambda \in [0,1].$$

Since $\mathcal{S}(-x,-\mu,0)=-\mathcal{S}(x,\mu,0)$ for $(x,\mu)\in C(J)\times\mathbb{R}$, $\mathcal{S}(\,\cdot\,,\,\cdot\,,0)$ is an odd operator. By Lemma 1, the restriction of $\mathcal{S}$ to $\overline{\Omega}\times[0,1]$ is a compact operator. Therefore, the Borsuk antipodal theorem and the homotopy property give [4]

$$\deg(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,0),\Omega,0)\neq 0,$$
$$\deg\big(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,0),\Omega,0\big)=\deg\big(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,1),\Omega,0\big),$$

where "deg" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identical operator on $C(J)\times\mathbb{R}$. Consequently, $\deg(\mathcal{I}-\mathcal{S}(\,\cdot\,,\,\cdot\,,1),\Omega,0)\neq 0$, which implies the existence of a fixed point of $\mathcal{S}(\,\cdot\,,\,\cdot\,,1)$. $\square$

Example 1. Let $\varphi,\psi,\gamma\in C(J)$ and $\varphi(t)\geq\varepsilon>0$, $\psi\geq 0$ on J. Then the function $f(t,x,y)=\varphi(t)(x+\sin y)+\psi(t)y+\gamma(t)$ satisfies condition (H_2) for $c=\|\gamma\|/\varepsilon+1$, $k=\|\varphi\|+\|\psi\|$ and $l=\|\varphi\|+\|\gamma\|$. If $a\in C(J)$, $a\geq 0$ on J and $a\not\equiv 0$, then Theorem 1 guarantees that the periodic problem

$$\left.\begin{array}{c} u'=a(t)^c\mathcal{D}^\alpha u+\varphi(t)\big(u+\sin({}^c\mathcal{D}^\beta u)\big)+\psi(t)^c\mathcal{D}^\beta u+\gamma(t),\\ u(0)=u(T)\end{array}\right\}$$

has at least one solution.

If $f(t,x,y)$ in (1) is independent of the variable y, that is, $f(t,x,y)=f(t,x)$, then Theorem 1 gives the following result for the periodic problem

$$\left.\begin{array}{c} u'(t)=a(t)^c\mathcal{D}^\alpha u(t)+f(t,u(t)),\\ u(0)=u(T).\end{array}\right\}\tag{4}$$

Corollary 1. *Let (H_1) hold and let $f\in C(J\times\mathbb{R})$ and there exist positive constants c, k and l such that*

$$f(t,x)<0\ \ \text{for }(t,x)\in J\times(-\infty,-c],\quad f(t,x)>0\ \ \text{for }(t,x)\in J\times[c,\infty),$$
$$|f(t,x)|\leq k|x|+l\ \ \text{for }(t,x)\in J\times\mathbb{R}.$$

Then the periodic problem (4) *has at least one solution.*

The following result gives the existence of a unique solution of problem (4).

Theorem 2. *Let the conditions of Corollary* 1 *be satisfies. In addition, suppose that $f(t,x)$ is increasing in x for all $t\in J$ and for any $\ell>0$ there exists $L_\ell>0$ such that*

$$|f(t,x)-f(t,y)|\leq L_\ell|x-y|\ \ \text{for }t\in J,\ x,y\in[-\ell,\ell].$$

Then the periodic problem (4) *has a unique solution.*

Example 2. Let $a,\varphi,\gamma\in C(J)$, $a\geq 0$, $\varphi(t)\geq\varepsilon>0$ on J and $a\not\equiv 0$. Then the periodic problem

$$\left.\begin{array}{c} u'=a(t)^c\mathcal{D}^\alpha u+\varphi(t)u+\gamma(t),\\ u(0)=u(T)\end{array}\right\}$$

has a unique solution.

References

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