To a Question on the Stability of Linear Hybrid Functional Differential Systems with Aftereffect

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1 Introduction

The recent general theory of functional differential equations [2]–[5] allowed us to give a clear and concise description of their basic properties including the properties of solution stability. At the same time broad classes of linear hybrid functional differential systems with aftereffect (LHFDSA) arising in many applications are not formally covered by the developed theory and remain out of view of specialists using functional differential and difference systems with aftereffect for simulation of real processes. Below we suggest hybrid functional differential analogues of fundamental assertions of the theory of functional differential equations for problems of stability.

2 The W-method of N. V. Azbelev

First, let us consider the case when one of the equations is a linear differential one and is defined on a set of discrete points, and the other one is a linear functional differential equation with aftereffect (LFDEA) on a semiaxis. For this case we describe the W-method scheme of N. V. Azbelev.

Let us denote the infinite matrix with the columns $y(-1), y(0), y(1), \dots, y(N), \dots$ of size n, by $y = \{y(-1), y(0), y(1), \dots, y(N), \dots\}$ and the infinite matrix with columns $g(0), g(1), \dots, g(N), \dots$ the of size n, by $g = \{g(0), g(1), \dots, g(N), \dots\}$.

Each infinite matrix
$$y = \big\{y(-1), y(0), y(1), \dots, y(N), \dots\big\}$$
can be associated with the vector function
$$y(t) = y(-1)\chi_{[-1,0)}(t) + y(0)\chi_{[0,1)}(t) + y(1)\chi_{[1,2)}(t) + \dots + y(N)\chi_{[N,N+1)}(t) + \cdots .$$

Similarly, each of the infinite matrices $g = \{g(0), g(1), \dots, g(N), \dots\}$ can be associated with the vector function
$$g(t) = g(0)\chi_{[0,1)}(t) + g(1)\chi_{[1,2)}(t) + \dots + g(N)\chi_{[N,N+1)}(t) + \cdots .$$

Let us denote the vector function $y(t) = y([t])$, $t \in [-1,\infty)$, by $y(t) = y[t]$ and the vector function $g(t) = g([t])$, $t \in [0,\infty)$, by $g[t]$.

The set of vector functions $y[\,\cdot\,]$ is denoted by ℓ_0. The set of vector functions $g[\,\cdot\,]$ is denoted by ℓ. Let $(\Delta y)(t) = y(t) - y(t-1) = y[t] - y[t-1]$ at $t \geq 1$, and $(\Delta y)(t) = y(t) = y[t] = y(0)$ at $t \in [0,1)$.

The abstract hybrid functional differential system takes the form
$$\begin{aligned} \mathcal{L}_{11}x + \mathcal{L}_{12}y = \dot{x} - F_{11}x - F_{12}y = f, \\ \mathcal{L}_{21}x + \mathcal{L}_{22}y = \Delta y - F_{21}x - F_{22}y = g. \end{aligned} \tag{1}$$

Here and below $\mathbb{R}^n$ is the space of vectors $\alpha = \operatorname{col}\{\alpha^1, \dots, \alpha^n\}$ with real components and the norm $\|\alpha\|_{\mathbb{R}^n}$. Assume the space L of locally summable $f : [0,\infty) \to \mathbb{R}^n$ with seminorms

$\|f\|_{L[0,T]} = \int\limits_0^T \|f(t)\|_{\mathbb{R}^n}\,dt$ for all the $T > 0$ and the space D of locally absolutely continuous functions $x : [0,\infty) \to \mathbb{R}^n$ with seminorms

$$\|x\|_{D[0,T]} = \|\dot{x}\|_{L[0,T]} + \|x(0)\|_{\mathbb{R}^n}$$

for all the $T > 0$.

Also assume the space ℓ of vector functions

$$g(t) = g(0)\chi_{[0,1)}(t) + g(1)\chi_{[1,2)}(t) + \cdots + g(N)\chi_{[N,N+1)}(t) + \cdots$$

with the seminorms $\|g\|_{\ell_T} = \sum\limits_{i=0}^{T} \|g_i\|_{\mathbb{R}^n}$ for all the $T \geq 0$ and the space ℓ_0 of vector functions

$$y(t) = y(-1)\chi_{[-1,0)}(t) + y(0)\chi_{[0,1)}(t) + y(1)\chi_{[1,2)}(t) + \cdots + y(N)\chi_{[N,N+1)}(t) + \cdots$$

with the seminorms $\|y\|_{\ell_{0T}} = \sum\limits_{i=-1}^{T} \|y_i\|_{\mathbb{R}^n}$ for all the $T \geq -1$.

The operators $\mathcal{L}_{11}, F_{11} : D \to L$, $\mathcal{L}_{12}, F_{12} : \ell_0 \to L$, $\mathcal{L}_{21}, F_{21} : D \to \ell$, $\mathcal{L}_{22}, F_{22} : \ell_0 \to \ell$ are assumed to be continuous linear and Volterra.

Let $\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$. Then (1) can be written as $\mathcal{L}\{x,y\} = \operatorname{col}\{f,g\}$. Suppose that for any $x(0) \in \mathbb{R}^n$ and $y(-1) \in \mathbb{R}^n$ the Cauchy problem for the "model" system $\dot{x} = F_{11}^0 x + F_{12}^0 z + z$, $\Delta y = F_{21}^0 z + F_{22}^0 y + u$, where the operators $F_{11}^0 : D \to L$, $F_{12}^0 : \ell_0 \to L$, $F_{12}^0 : \ell_0 \to L$, $F_{21}^0 : D \to \ell$, $F_{22}^0 : \ell_0 \to \ell$ are assumed to be continuous linear and Volterra. Then the model system can be written as $\mathcal{L}_0\{x,y\} = \operatorname{col}\{z,u\}$. Suppose its solution can be represented as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x(0) \\ y(-1) \end{pmatrix} + \begin{pmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix}.$$

Here $\mathcal{W} : L \times \ell \to D \times \ell_0$ is the continuous Volterra operator, the Cauchy operator for the system, $\mathcal{W} = \begin{pmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{pmatrix}$, $U : \mathrm{R}^n \times \mathbb{R}^n \to D \times \ell_0$ is the fundamental matrix for the system $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$.

If the elements $\operatorname{col}\{x,y\} : [0,\infty) \times [-1,\infty) \to \mathbb{R}^n \times \mathbb{R}^n$ forming the Banach space $\mathbf{D} \times \mathbf{M}_0 \cong (\mathbf{B} \times \mathbb{R}^n) \times (\mathbf{M} \times \mathbb{R}^n)$ (space $\mathbf{D} \subset D$, space $\mathbf{M}_0 \cong \mathbf{M} \oplus \mathbb{R}^n \subset \ell_0$, space $\mathbf{B} \subset L$, space $\mathbf{M} \subset \ell$, $\mathbf{B}, \mathbf{M}$ are the Banach spaces) have certain specific properties, such as

$$\sup_{t \geq 0} \|x(t)\|_{\mathbb{R}^n} + \sup_{k=-1,0,1,\dots} \|y(k)\|_{\mathbb{R}^n} < \infty,$$

and the Cauchy problem is uniquely solvable for the equation $\mathcal{L}\{x,y\} = \operatorname{col}\{f,g\}$ with the bounded linear operator $\mathcal{L} : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$, then the solutions of this problem have the same asymptotic properties. This follows from the theorem given below [6] (see [2, Theorem 2.1.1] and [1, Theorem 1]).

Theorem. *Assume $\mathcal{W} : \mathbf{B} \times \mathbf{M} \to \mathbf{D} \times \mathbf{M}_0$ is the bounded Cauchy operator of the Cauchy problem for the model equation $\mathcal{L}_0\{x,y\} = \operatorname{col}\{f,g\}$, $\operatorname{col}\{x(0), y(-1)\} = \operatorname{col}\{0,0\}$ and U is the fundamental matrix of the model equation $\mathcal{L}_0\{x,y\} = \operatorname{col}\{0,0\}$. Here $\mathcal{L}_0 : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$. Assume the linear operator $\mathcal{L} : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ is bounded, C is the Cauchy operator of the Cauchy problem $\mathcal{L}\{x,y\} = \operatorname{col}\{f,g\}$, $\operatorname{col}\{x(0), y(-1)\} = \operatorname{col}\{0,0\}$ and X is the fundamental matrix of the equation $\mathcal{L}\{x,y\} = \operatorname{col}\{0,0\}$. Then for the equality*

$$\mathcal{W}\{\mathbf{B}, \mathbf{M}\} + U\{\mathbb{R}^n, \mathbb{R}^n\} = C\{\mathbf{B}, \mathbf{M}\} + X\{\mathbb{R}^n, \mathbb{R}^n\} \tag{2}$$

to hold true it is necessary and sufficient that the operator $\mathcal{L}\mathcal{W}$ (the operator $\mathcal{W}\mathcal{L}$) have a bounded inverse

$$(\mathcal{L}\mathcal{W})^{-1} : \mathbf{B} \times \mathbf{M} \to \mathbf{B} \times \mathbf{M} \quad \big((\mathcal{W}\mathcal{L})^{-1} : (\mathbf{D} \times \mathbf{M}_0)^0 \to (\mathbf{D} \times \mathbf{M}_0)^0 \big),$$

where $(\mathbf{D} \times \mathbf{M}_0)^0 = \{\operatorname{col}\{x, y\} \in \mathbf{D} \times \mathbf{M}_0 : \ \operatorname{col}\{x(0), y(-1)\} = \operatorname{col}\{0, 0\}\}$.

Corollary ([1], [2, pp. 36, 48])**.** *If the operator $\mathcal{L} : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ is bounded and $\|(\mathcal{L} - \mathcal{L}_0)\mathcal{W}\|_{\mathbf{B}\times\mathbf{M}\to\mathbf{B}\times\mathbf{M}} < 1$ is true or $\|\mathcal{W}(\mathcal{L} - \mathcal{L}_0)\|_{(\mathbf{D}\times\mathbf{M}_0)^0\to(\mathbf{D}\times\mathbf{M}_0)^0} < 1$ is true, then Equality* (2) *holds true as well.*

In the case when (2) holds true (when the solution spaces of the model equation and equation under study coincide), we say that the equation $\mathcal{L}\{x, y\} = \operatorname{col}\{f, g\}$ has the property $\mathbf{D} \times \mathbf{M}_0$, or, in short, the equation is $\mathbf{D} \times \mathbf{M}_0$-stable.

Assume the model equation [1]–[5] and Banach space $\mathbf{B}$ with the elements of the space L ($\mathbf{B} \subset L$, this embedding is continuous) are selected so that the solutions of this equation possess asymptotic properties we are interested in.

We introduce the Banach space $D(\mathcal{L}_{11}, \mathbf{B})$ with the norm

$$\|x\|_{D(\mathcal{L}_{11},\mathbf{B})} = \|\mathcal{L}_{11}x\|_{\mathbf{B}} + \|x(0)\|_{\mathbb{R}^n}.$$

Assume that the operator $\mathcal{W}_{11}$ acts continuously from the space $\mathbf{B}$ into the space $\mathbf{B}$, and the operator U_{11} acts from space $\mathbb{R}^n$ into the space $\mathbf{B}$. This condition is equivalent to the fact [1]–[5] that the space $D(\mathcal{L}_{11}, \mathbf{B})$ is linearly isomorphic to the Sobolev space with the norm

$$\|x\|_{W^{(1)}_{\mathbf{B}}[0,\infty)} = \|\dot{x}\|_{\mathbf{B}} + \|x\|_{\mathbf{B}}.$$

Hereinafter this space is referred to as $W_{\mathbf{B}}$ ($W_{\mathbf{B}} \subset D$, this embedding is continuous).

The equation $\mathcal{L}_{11}x = z$ with the operator $\mathcal{L}_{11} : W_{\mathbf{B}} \to \mathbf{B}$ is $D(\mathcal{L}_{11}, \mathbf{B})$-stable if and only if it is strongly $\mathbf{B}$-stable. $\mathcal{L}_{11}x = z$ is strongly $\mathbf{B}$- stable if for any $z \in \mathbf{B}$ each solution x of this equation has the property $x \in \mathbf{B}$ and $\dot{x} \in \mathbf{B}$ ([2, Ch. IV, § 4.6], [5]).

3 Reduction of LFDEA on the Semiaxis

Let us consider the scheme from Clause 2 for two equations (1). The operators $\mathcal{L}_{11} : D \to L$, $\mathcal{L}_{12} : \ell_0 \to L$, $\mathcal{L}_{21} : D \to \ell$, $\mathcal{L}_{22} : \ell_0 \to \ell$ are considered as reduction to pairs $(\mathbf{W_B}, \mathbf{B})$, $(\mathbf{M_0}, \mathbf{B})$, $(\mathbf{W_B}, \mathbf{M})$, $(\mathbf{M_0}, \mathbf{M})$. These operators are assumed to be Volterra linear and bounded operators.

Assume that the general solution of the equation $\mathcal{L}_{22}y = g$ for $g \in M$is the space of M_0 and is represented by the Cauchy formula

$$y[t] = Y_{22}[t]y(-1) + \sum_{s=0}^{[t]} C_{22}[t, s]g[s].$$

Let

$$(C_{22}g)[t] = \sum_{s=0}^{[t]} C_{22}[t, s]g[s], \quad (Y_{22}y(-1))[t] = Y_{22}[t]y(-1).$$

Then every solution y of the second equation in (1) has the form

$$y = -C_{22}\mathcal{L}_{21}x + Y_{22}y(-1) + C_{22}g.$$

Substituting the first equation into (1), we obtain

$$\mathcal{L}_{11}x + \mathcal{L}_{12}y = \mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x + \mathcal{L}_{12}Y_{22}y(-1) + \mathcal{L}_{12}C_{22}g = f,$$
$$\mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x = f_1 = f - \mathcal{L}_{12}Y_{22}y(-1) - \mathcal{L}_{12}C_{22}g.$$

Let $\mathcal{L} = \mathcal{L}_{11} - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}$, then the first equation in (1) takes the form of $\mathcal{L}x = f_1$. Suppose the Volterra operator $\mathcal{L} : (\mathbf{W_B})^0 \to B$ is Volterra invertible, that is (when the Cauchy problem for $\mathcal{L}x = f_1$ possesses the following property: at any $f_1 \in \mathbf{B}$ its solutions are $x \in \mathbf{W_B}$). Thus, we solved the problem, when for Equation (1) at any $\{f, g\} \in \mathbf{B} \times \mathbf{M}$ its solutions are $\{x, y\} \in \mathbf{W_B} \times \mathbf{M}$.

4 Reduction to a Linear Difference Equation (LDE) on a Discrete Set of Points

Let us use the ability of the hybrid system to be reduced to a LDE defined on a discrete set of points. For Equation (1) we use the designations given in Clauses 2 and 3.

Assume the general solution of the equation $\mathcal{L}_{11}x = f$ for $f \in L$ is a member of the space D and is represented by the Cauchy formula

$$x(t) = X_{11}(t)x(0) + \int_0^t C_{11}(t,s)f(s)\,ds.$$

Let $(C_{11}f)(t) = \int_0^t C_{11}(t,s)f(s)\,ds$, $(X_{11}x(0))(t) = X_{11}(t)x(0)$, then for $x \in D$ the representation $x = X_{11}x(0) + C_{11}f$ holds true.

The first variable x can be estimated out of the first equation in (1)

$$x = -C_{11}\mathcal{L}_{12}y + X_{11}x(0) + C_{11}f.$$

We use this substitution in the second equation of (1), we obtain

$$\begin{gathered}\mathcal{L}_{21}x + \mathcal{L}_{22}y = -\mathcal{L}_{21}C_{11}\mathcal{L}_{12}y + \mathcal{L}_{21}X_{11}x(0) + \mathcal{L}_{21}C_{11}f + \mathcal{L}_{22}y = g,\\ -\mathcal{L}_{21}C_{11}\mathcal{L}_{12}y + \mathcal{L}_{22}y = g_1 = g - \mathcal{L}_{21}X_{11}x(0) - \mathcal{L}_{21}C_{11}f.\end{gathered}$$

Let $\mathcal{L} = \mathcal{L}_{22} - \mathcal{L}_{21}C_{11}\mathcal{L}_{12}$, then the second equation in (1) takes the form $\mathcal{L}y = g_1$. Suppose the Volterra operator $\mathcal{L} : (\mathbf{M_0})^0 \to \mathbf{M}$ is Volterra invertible (when the Cauchy problem for $\mathcal{L}y = g_1$ at any $g_1 \in \mathbf{M}$ its solutions are $x \in \mathbf{M}_0$). Thus, we solved the problem, when at any $\{f,g\} \in \mathbf{B} \times \mathbf{M}$ for (1) its solutions are $\{x,y\} \in \mathbf{D} \times \mathbf{M_0}$.

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