How to Construct Solutions of State-Dependent Impulsive Boundary Value Problems

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1 Formulation of the Problem

We consider the nonlinear system of differential equations

$$u'(t) = f(t, u(t)), \ \text{ a.e. } t \in [a, b] \subset \mathbb{R}, \tag{1}$$

with continuous $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$. Equation (1) is subject to the *state-dependent* impulse condition

$$u(t+) - u(t-) = \gamma(u(t-)) \ \text{ for such } t \text{ that } \ g(t, u(t-)) = 0. \tag{2}$$

Here $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ are continuous, and the impulse instants $t \in (a, b)$ in (2) are unknown. These instants are called state-dependent because they depend on a solution u through the equation $g(t, u(t-)) = 0$. Impulsive problem (1), (2) is investigated together with the linear boundary condition

$$Au(a) + Cu(b) = d, \tag{3}$$

where d is a constant vector, and A, C are constant (possibly singular) matrices satisfying the condition $\operatorname{rank}[A, C] = n$.

A left-continuous vector-function $u : [a, b] \to \mathbb{R}^n$ is called a *solution* of problem (1)–(3) if there exist $p \in \mathbb{N}$ and $t_i \in (a, b)$, $i = 1, \dots, p$, such that:

- $a < t_1 < t_2 < \dots < t_p < b$,
- the restrictions $u|_{[a,t_1]}, u|_{(t_1,t_2]}, \dots, u|_{(t_p,b]}$ have continuous derivatives,
- u satisfies (1) for $t \in [a, b]$, $t \neq t_i$, $i = 1, \dots, p$,
- u satisfies (2) for $t = t_i$, i.e. $u(t_i+) - u(t_i) = \gamma(u(t_i))$, $g(t_i, u(t_i)) = 0$, $i = 1, \dots, p$,
- u fufils the boundary conditions (3).

The set

$$G = \big\{(t, x) \in [a, b] \times \mathbb{R}^n : \ g(t, x) = 0\big\} \tag{4}$$

is called a *barrier*.

Wee see that if u satisfies condition (2) for $t = t_i \in (a, b)$, then u has an intersection point $(t_i, u(t_i))$ with the barrier G, and in addition, u has a jump of the size $\gamma(u(t_i))$ at the point t_i.

Most of the results in the literature devoted to boundary value problems concern fixed-times impulses. A reason for the lack of results for state-dependent impulsive boundary value problems lies in the fact that state-dependent impulses significantly change properties of boundary value problems. In the book [2], state dependent impulsive boundary value problems with barriers given explicitely in the form $t = g(x)$ are investigated. The existence results in [2] are reached by means of fixed point theorems or topological degree methods. But there are no constructive numerical results for state-dependent impulsive boundary value problems in the literature. This is our main motivation for the investigation of problem (1)–(3).

We focus our attention to the case where $p = 1$, that is u has a unique intersection point with the barrier G, and then we use the technique suggested in [3], which makes it possible to discuss the solvability of problem (1)–(3) as well as to find approximate solutions. This approach is based on a construction of two simple parametrized model problems (5), (6) and (7), (8). We give conditions which guarantee that if the parameters τ, ξ, λ, η belong to some bounded sets, then solutions of these parametrized model problems can be obtained as limits of uniformly convergent sequences of successive approximations (10) and (12). Equations in the parametrized model problems contain functional perturbation terms which essentially depend on the parameters and which together with the original boundary conditions (3) and the barrier (4) generate the system of algebraic *determining* equations (14). Numerical values of the parameters should be found from (14) in the bounded sets mentioned above where the uniform convergence is guaranteed. A solution of problem (1)–(3) is then constructed (see (13)) by means of such solutions of problems (5), (6) and (7), (8) which have the values of parameters satisfying (14). Consequently, the infinite-dimensional problem (1)–(3) is reduced to the finite-dimensional algebraic system (14).

In practice, we investigate system (14), where explicitly determined successive approximations are written instead of their limits (cf. (16)). Then the solvability of (14) can be checked more easily and we get approximate solutions of problem (1)–(3) and error estimates using for example Maple 14. By our knowledge this is the first numerical-analytic method for this type of impulsive problems. This method can be applied on problems with linear as well as with nonlinear boundary conditions.

2 Construction of Solutions

Choose a compact convex set $\Omega_a \subset \mathbb{R}^n$ and put $\Omega_b = \{x + \gamma(x) : x \in \Omega_a\}$. Consider a scalar parameter $\tau \in (a, b)$ together with vector parameters $\xi, \lambda \in \Omega_a$ and $\eta \in \Omega_b$. Instead of the impulsive boundary value problem (1)–(3) we study two auxiliary parametrized boundary value problems on the intervals $[a, \tau]$ and $[\tau, b]$, respectively:

$$x'(t) = f(t, x(t)) + \frac{1}{\tau - a}\Big(\lambda - \xi - \int\limits_a^\tau f(s, x(s))\,\mathrm{d}s\Big), \tag{5}$$

$$x(a) = \xi, \quad x(\tau) = \lambda, \tag{6}$$

and

$$y'(t) = f(t, y(t)) + \frac{1}{b - \tau}\Big(\eta - (\lambda + \gamma(\lambda)) - \int\limits_\tau^b f(s, y(s))\,\mathrm{d}s\Big), \tag{7}$$

$$y(\tau) = \lambda + \gamma(\lambda), \quad y(b) = \eta. \tag{8}$$

I. Let us connect problem (5), (6) with the parametrized sequence of functions

$$x_0(t; \tau, \xi, \lambda) = \Big(1 - \frac{t - a}{\tau - a}\Big)\xi + \frac{t - a}{\tau - a}\,\lambda, \ \ t \in [a, \tau], \tag{9}$$

$$\begin{aligned} x_m(t; \tau, \xi, \lambda) = \xi + \int\limits_a^t f\big(s, x_{m-1}(s; \tau, \xi, \lambda)\big)\,\mathrm{d}s - \\ - \frac{t - a}{\tau - a}\int\limits_a^\tau f\big(s, x_{m-1}(s; \tau, \xi, \lambda)\big)\,\mathrm{d}s + \frac{t - a}{\tau - a}\,(\lambda - \xi), \ \ t \in [a, \tau], \ \ m \in \mathbb{N}. \end{aligned} \tag{10}$$

II. Let us connect problem (7), (8) with the parametrized sequence of functions

$$y_0(t; \tau, \lambda, \eta) = \Big(1 - \frac{t - \tau}{b - \tau}\Big)(\lambda + \gamma(\lambda)) + \frac{t - \tau}{b - \tau}\,\eta, \ \ t \in [\tau, b], \tag{11}$$

$$
\begin{aligned}
y_m(t;\tau,\lambda,\eta) = (\lambda+\gamma(\lambda)) + \int_\tau^t f\big(s, y_{m-1}(s;\tau,\lambda,\eta)\big)\,\mathrm{d}s - \\
- \frac{t-\tau}{b-\tau}\int_\tau^b f\big(s, y_{m-1}(s;\tau,\lambda,\eta)\big)\,\mathrm{d}s + \\
+ \frac{t-\tau}{b-\tau}\big(\eta-(\lambda+\gamma(\lambda))\big), \ \ t\in[\tau,b], \ \ m\in\mathbb{N}. \qquad (12)
\end{aligned}
$$

Choose $\rho\in\mathbb{R}^n$ and assume that $\mathcal{O}_a\subset\mathbb{R}^n$ and $\mathcal{O}_b\subset\mathbb{R}^n$ are componentwise neighbourhoods of Ω_a and Ω_b, respectively. We have proved that if f fulfils the Lipschitz conditions $|f(t,x)-f(t,y)|\le K|x-y|$ on $\mathcal{O}_a$ and $\mathcal{O}_b$ with a sufficiently large vector ρ and with a sufficiently small matrix K, then

$$
\lim_{m\to\infty} x_m(t;\tau,\xi,\lambda) = x_\infty(t;\tau,\xi,\lambda) \ \text{ uniformly on } \ [a,\tau],
$$

and

$$
\lim_{m\to\infty} y_m(t;\tau,\lambda,\eta) = y_\infty(t;\tau,\lambda,\eta) \ \text{ uniformly on } \ [\tau,b].
$$

More precisely, on $\mathcal{O}_a$:

$$
\rho \ge \frac{b-a}{4}\,\delta_{\mathcal{O}_a}(f), \quad r(K) < \frac{10}{3(b-a)}\,,
$$

where

$$
\delta_{\mathcal{O}_a}(f) =: \max_{[a,b]\times\mathcal{O}_a} f(t,x) - \min_{[a,b]\times\mathcal{O}_a} f(t,x),
$$

and $r(K)$ is the spectral radius of K. Similarly, on $\mathcal{O}_b$.

Further, we have proved that for each $\tau\in(a,b)$, $\xi,\lambda\in\Omega_a$, the vector function $x_\infty(t;\tau,\xi,\lambda)$ is a unique solution of problem (5), (6) and that for each $\tau\in(a,b)$, $\lambda\in\Omega_a$, $\eta\in\Omega_b$, the vector function $y_\infty(t;\tau,\lambda,\eta)$ is a unique solution of problem (7), (8).

Finally, we have found such values of the parameters τ, ξ, λ, η that the solution u of (1)–(3) can be written in the form

$$
u(t) = \begin{cases} x_\infty(t;\tau,\xi,\lambda) & \text{if } \ t\in[a,\tau], \\ y_\infty(t;\tau,\lambda,\eta) & \text{if } \ t\in(\tau,b]. \end{cases} \qquad (13)
$$

It turned out that such parameters τ,ξ,λ,η fulfil the system of algebraic "determining" equations

$$
\begin{cases}
\lambda-\xi-\displaystyle\int_a^\tau f\big(s, x_\infty(s;\tau,\xi,\lambda)\big)\,\mathrm{d}s = 0, \\
\big(\eta-(\lambda+\gamma(\lambda))\big) - \displaystyle\int_\tau^b f\big(s, y_\infty(s;\tau,\lambda,\eta)\big)\,\mathrm{d}s = 0, \\
A\xi + C\eta = d, \\
g(\tau,\lambda) = 0,
\end{cases} \qquad (14)
$$

and in addition,

$$
g\big(t, y_\infty(t;\tau,\lambda,\eta)\big) \ne 0, \ \ t\in(\tau,b]. \qquad (15)
$$

The solvability of the determining system (14) can be established by studying its approximate version

$$\begin{cases} \lambda - \xi - \displaystyle\int_a^\tau f\big(s, x_m(s;\tau,\xi,\lambda)\big)\,\mathrm{d}s = 0, \\ \big(\eta - (\lambda + \gamma(\lambda))\big) - \displaystyle\int_\tau^b f\big(s, y_m(s;\tau,\lambda,\eta)\big)\,\mathrm{d}s = 0, \\ A\tau + C\eta = d, \\ g(\tau,\lambda) = 0, \end{cases} \tag{16}$$

with

$$g\big(t, y_m(t;\tau,\lambda,\eta)\big) \neq 0, \quad t \in (\tau, b], \tag{17}$$

which can be constructed explicitely for a fixed m. System (16) can be solved, for example, by Maple 14. If the quartet $(\widehat{\tau}, \widehat{\xi}, \widehat{\lambda}, \widehat{\eta}) \in (a,b) \times \Omega_a \times \Omega_a \times \Omega_b$ is a root of system (16) and inequality (17) holds, then $\widehat{\xi}$ is an approximation of the inital value $u(a)$ of the solution u of problem (1)–(3), $\widehat{\tau}$ is an approximation of the impulse point τ of u, $\widehat{\lambda}$ is an approximation of $u(\tau)$ and $\widehat{\lambda} + \gamma(\widehat{\lambda})$ is an approximation of $u(\tau+)$.

References

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