Oscillation Criteria for Certain System of Non-Linear Ordinary Differential Equations

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On the half-line $\mathbb{R}_+ = [0, +\infty[$, we consider the two-dimensional system of nonlinear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}} \operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha} \operatorname{sgn} u, \end{aligned} \tag{1}$$

where $\alpha > 0$ and $p, g : \mathbb{R}_+ \to \mathbb{R}$ are locally Lebesgue integrable functions.

By a solution of system (1) on the interval $J \subseteq [0, +\infty[$ we understand a pair (u, v) of functions $u, v : J \to \mathbb{R}$, which are absolutely continuous on every compact interval contained in J and satisfy equalities (1) almost everywhere in J.

It was proved by Mirzov in [10] that all non-extendable solutions of system (1) are defined on the whole interval $[0, +\infty[$. Therefore, when we are speaking about a solution of system (1), we assume that it is defined on $[0, +\infty[$.

Definition 1. A solution (u, v) of system (1) is called *non-trivial* if $u \not\equiv 0$ on any neighborhood of $+\infty$. We say that a non-trivial solution (u, v) of system (1) is *oscillatory* if the function u has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

In [10, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (1), if the additional assumption

$$g(t) \geq 0 \ \text{ for a.e. } \ t \geq 0 \tag{2}$$

is satisfied. Especially, under assumption (2), if system (1) has an oscillatory solution, then any other its non-trivial solution is also oscillatory.

On the other hand, it is clear that if $g \equiv 0$ on some neighborhood of $+\infty$, then all non-trivial solutions of system (1) are non-oscillatory. That is why it is natural to assume that inequality (2) is satisfied and

$$\operatorname{meas}\big\{\tau \geq t : \ g(\tau) > 0\big\} > 0 \ \text{ for } \ t \geq 0. \tag{3}$$

Definition 2. We say that system (1) is *oscillatory* if all its non-trivial solutions are oscillatory.

Oscillation theory for ordinary differential equations and their systems is a widely studied and well-developed topic of the qualitative theory of differential equations. As for the results which are closely related to those of this section, we should mention [2, 4, 5, 6, 7, 8, 9, 11, 12, 13]. Some criteria established in these papers for the second order linear differential equations or for two-dimensional systems of linear differential equations are generalized to the considered system (1) below.

Many results (see, e.g., survey given in [2]) have been obtained in oscillation theory of so-called "half-linear" equation

$$\big(r(t)|u'|^{q-1} \operatorname{sgn} u'\big)' + p(t)|u|^{q-1} \operatorname{sgn} u = 0 \tag{4}$$

(alternatively this equation is referred as "equation with the scalar q-Laplacian"). Equation (4) is usually considered under the assumptions $q > 1$, $p, r : [0, +\infty[\to \mathbb{R}$ are continuous and r is positive. One can see that equation (4) is a particular case of system (1). Indeed, if the function u, with properties $u \in C^1$ and $r|u'|^{q-1} \operatorname{sgn} u' \in C^1$, is a solution of equation (4), then the vector function $(u, r|u'|^{q-1} \operatorname{sgn} u')$ is a solution of system (1) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q - 1$.

Moreover, the equation

$$u'' + \frac{1}{\alpha}\,p(t)|u|^{\alpha}|u'|^{1-\alpha}\operatorname{sgn}u = 0 \tag{5}$$

is also studied in the existing literature under the assumptions $\alpha \in]0,1]$ and $p:\mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. It is mentioned in [6] that if u is a so-called proper solution of (5) then it is also a solution of system (1) with $g \equiv 1$ and vice versa. Some oscillations and non-oscillations criteria for equation (5) can be found, e.g., in [6, 7].

Finally, we mention the paper [1], where a certain analogy of Hartman–Wintner's theorem is established (origin one can find in [3, 14]), which allows us to derive oscillation criteria of Hille's type for system (1).

In what follows, we assume that the coefficient g is non-integrable on $[0,+\infty[$, i.e.,

$$\int\limits_0^{+\infty} g(s)\,ds = +\infty. \tag{6}$$

Let

$$f(t) := \int\limits_0^{t} g(t)\,ds \ \text{ for } \ t \geq 0.$$

In view of assumptions (2), (3), and (6), we have

$$\lim_{t\to+\infty} f(t) = +\infty$$

and there exists $t_g \geq 0$ such that $f(t) > 0$ for $t > t_g$ and $f(t_g) = 0$. We can assume without loss of generality that $t_g = 0$, since we are interested in behaviour of solutions in the neighbourhood of $+\infty$, i.e., we have

$$f(t) > 0 \ \text{ for } \ t > 0.$$

We put

$$c_\alpha(t) := \frac{\alpha}{f^{\alpha}(t)} \int\limits_0^{t} \frac{g(s)}{f^{1-\alpha}(s)} \bigg(\int\limits_0^{s} (\xi)p(\xi)\,d\xi \bigg)\,ds \ \text{ for } \ t > 0.$$

Now, we formulate an analogue (in a suitable form for us) of the Hartman–Wintner's theorem for the system (1) established in [1].

Theorem 3 ([1, Corollary 2.5 (with $\nu = 1-\alpha$)])**.** *Let conditions* (2), (3), *and* (6) *hold, and either*

$$\lim_{t\to+\infty} c_\alpha(t) = +\infty,$$

or

$$-\infty < \liminf_{t\to+\infty} c_\alpha(t) < \limsup_{t\to+\infty} c_\alpha(t).$$

Then system (1) *is oscillatory.*

One can see that two cases are not covered by Theorem 3, namely, the function $c_\alpha(t)$ has a finite limit and $\liminf\limits_{t\to+\infty} c_\alpha(t) = -\infty$. Our aim is to find oscillation criteria for system (1) in the first mentioned case. Consequently, in what follows, we assume that

$$\lim_{t\to+\infty} c_\alpha(t) =: c_\alpha^* \in \mathbb{R}. \tag{7}$$

Now we formulate main results.

Theorem 4. *Let* (7) *and the inequality*

$$\limsup_{t\to+\infty} \frac{f^\alpha(t)}{\ln f(t)}\,(c_\alpha^* - c_\alpha(t)) > \Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}$$

hold. Then system (1) *is oscillatory.*

For better formulation of the next statement we introduce the following notations.

$$Q(t;\alpha) := f^\alpha(t)\bigg(c_\alpha^* - \int\limits_0^t p(s)(s)\,ds\bigg) \ \text{ for } \ t>0,$$

where the number c_α^* is given by (7). Moreover, we denote lower and upper limits of the function $Q(\,\cdot\,;\alpha)$ as follows

$$Q_*(\alpha) := \liminf_{t\to+\infty} Q(t;\alpha), \quad Q^*(\alpha) := \limsup_{t\to+\infty} Q(t;\alpha).$$

Oscillation criteria from the next theorem coincide with the well-known Hille's results for the second order linear differential equations established in [4].

Theorem 5. *Let* (7) *hold. Let, moreover, either*

$$Q_*(\alpha) > \frac{1}{\alpha}\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha},$$

or

$$Q^*(\alpha) > 1.$$

Then system (1) *is oscillatory.*

Remark 6. Presented statements generalize results stated in [2, 4, 5, 6, 7, 8, 9, 11, 13] concerning system (1) as well as equations (4) and (5). In particular, if we put $\alpha=1$, then we obtain oscillatory criteria for linear system of differential equations presented in [13]. Moreover, the results of [6] obtained for equation (5) are in a compliance with those above, where we put $g\equiv 1$. Observe also that Theorem 5 extends oscillation criteria for equation (5) stated in [7], where the coefficient p is suppose to be non-negative. In the monograph [2], it is noted that the assumption $p(t)\geq 0$ for t large enough can be easily relaxed to $\int\limits_0^t p(s)ds > 0$ for large t. It is worth mentioning here that we do not require any assumption of this kind.

Finally we show an example, where we can not apply oscillatory criteria from the above mentioned papers, but we can use Theorem 4 succesfully.

Example 7. Let $\alpha=2$, $g(t)\equiv 1$, and

$$p(t) := t\cos\Big(\frac{t^2}{2}\Big) + \frac{1}{(t+1)^3} \ \text{ for } \ t\geq 0.$$

It is clear that the function p and its integral

$$\int\limits_0^t p(s)\,ds = \sin\Big(\frac{t^2}{2}\Big) - \frac{1}{2(t+1)^2} + \frac{1}{2} \ \text{ for } \ t\geq 0$$

change their sign in any neighbourhood of $+\infty$. Therefore neither of the results mentioned in Remark 6 can be applied.

On the other hand, we have

$$
\begin{aligned}
c_2(t) &= \frac{2}{t^2}\int_0^t s\bigg(\int_0^s \Big(\xi\cos\frac{\xi^2}{2}+\frac{1}{(\xi+1)^3}\Big)\,d\xi\bigg)\,ds \\
&= \frac{1}{2}-\frac{2\cos\frac{t^2}{2}}{t^2}+\frac{3}{t^2}-\frac{\ln(t+1)}{t^2}-\frac{1}{t^2(t+1)} \ \text{ for } \ t>0
\end{aligned}
$$

and thus, the function $c_2(\,\cdot\,)$ has the finite limit

$$
c_\alpha^* = \lim_{t\to+\infty} c_2(t) = \frac{1}{2}\,.
$$

Moreover,

$$
\limsup_{t\to+\infty}\frac{t^2}{\ln t}\,(c_\alpha^*-c_2(t)) = \limsup_{t\to+\infty}\Big(\frac{2\cos\frac{t^2}{2}-3}{\ln t}+\frac{\ln(t+1)}{\ln t}+\frac{1}{(t+1)\ln t}\Big) = 1.
$$

Consequently, according to Theorem 4, system (1) is oscillatory.

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