On Integral Conditions Determining Some Γ-Ultimate Classes of Perturbations

E. K. Makarov

Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: jcm@im.bas-net.by

Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+ := [0, +\infty[\,, \tag{1}$$

with a piecewise continuous bounded coefficient matrix A and with the Cauchy matrix X_A. Together with system (1), consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+, \tag{2}$$

with a piecewise continuous bounded perturbation matrix Q. For the higher exponent of system (2), we use the notation $\lambda_n(A+Q)$. By $\mathbb{R}^{n\times n}$ we denote the set of all real $n\times n$-matrices with the spectral norm $\|\cdot\|$. By $\mathrm{PC}_n(\mathbb{R}^+)$ we denote the linear space of all piecewise continuous matrix functions $S : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$. The space of bounded elements of $\mathrm{PC}_n(\mathbb{R}^+)$ is denoted by $\mathrm{KC}_n(\mathbb{R}^+)$. Lyapunov exponent of $\beta \in \mathrm{PC}_1(\mathbb{R}^+)$ is denoted by $\lambda[\beta]$. We say that a function $\gamma \in \mathrm{PC}_1(\mathbb{R}^+)$ is strictly positive iff the condition $\inf\limits_{t\in J} \gamma(t) > 0$ holds for every finite interval $J \subset \mathbb{R}^+$.

Let $\mathfrak{M}$ be a class of perturbations. It is well known that the number $\Lambda(\mathfrak{M}) := \sup\{\lambda_n(A+Q) : Q \in \mathfrak{M}\}$ is an important asymptotic characteristics for system (1) [1, p. 157], [2, p. 39]. Many authors investigated how to find $\Lambda(\mathfrak{M})$ for various $\mathfrak{M}$ (see, e.g. [3]– [13]). In numerous cases, an algorithm similar to the algorithm for the computation of the sigma-exponent [3] can be constructed for $\Lambda(\mathfrak{M})$. In some other cases [4], [5], [10]– [13], the result is similar to the formula

$$\Omega(A) = \lim_{T\to+\infty} \varlimsup_{m\to\infty} \frac{1}{mT} \sum_{k=1}^{m} \ln \|X_A(kT, kT-T)\|$$

for the computation of the central exponent [1, p. 99], [10].

Let $\mathbb{T}$ be the set of all sequences $\tau : \mathbb{N}_0 \to \mathbb{R}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, monotonically increasing to $+\infty$. For arbitrary $\tau \in \mathbb{T}$, let

$$\Omega(A,\tau) = \varlimsup_{k\to\infty} \frac{1}{t_{k+1}} \sum_{i=0}^{k} \ln \|X_A(t_{i+1}, t_i)\|,$$

where $t_i := \tau(i), i \in \mathbb{N}_0$.

Definition 1. A class of perturbations $\mathfrak{M}$ is called Γ-ultimate if there exists a set $\Gamma \subset \mathbb{T}$ such that the relation

$$\Lambda(\mathfrak{M}) = \sup_{\tau\in\Gamma} \Omega(A,\tau)$$

is valid for every system (1).

In [14] we give sufficient conditions for $\mathfrak{M}$ to be Γ-ultimate when $\mathfrak{M}$ is defined by some conditions of the form $\|Q(t)\| \le N\beta(t)$, where $N > 0$ and β is taken from a certain family $\mathcal{K} \subset \mathrm{KC}_1(\mathbb{R}^+)$.

In the report we present an analogous condition for classes of perturbations $\mathfrak{N}_n[\mathcal{P}] \subset \mathrm{KC}_n(\mathbb{R}^+)$ defined by integral conditions. More precisely, by $\mathfrak{N}_n[\mathcal{P}]$ we denote the set of perturbations $Q \in \mathrm{KC}_n(\mathbb{R}^+)$ such that Q satisfies the condition

$$\lim_{t\to+\infty} t^{-1}\int_0^t p(s)\|Q(s)\|\,ds = 0$$

for some $p \in \mathcal{P}$, where $\mathcal{P} \subset \mathrm{PC}_1(\mathbb{R}^+)$ is a given set of nonnegative functions. In what follows, we refer to $\mathcal{P}$ as a collection of weights.

For each $\tau \in \mathbb{T}$ and $N > 0$, define the function $K_N^\tau : \mathbb{R}^+ \to \mathbb{R}$ by $K_N^\tau(s) = e^{N(s-t_k)}$ for $s \in]t_k, t_{k+1}]$, $k \in \mathbb{N}$, and $K_N^\tau(s) = 0$ for $s \le t_0$, where $t_k := \tau(k)$, $k \in \mathbb{N}_0$, are the elements of the sequence τ. Let us also put

$$\gamma(\beta,\tau) = \overline{\lim_{k\to\infty}}\,\frac{1}{t_{k+1}}\sum_{i=m_0}^{k}\ln\frac{2}{\sin\varphi_i},\quad \varphi_i = \min\Big\{\frac{\pi}{2}, e^{-2N_A}\int_{t_i-1}^{t_i}\beta(s)ds\Big\},\ \ i \ge m_\tau,$$

where $\tau \in \mathbb{T}$, $\beta \in \mathrm{KC}_1(\mathbb{R}^+)$, $m_\tau := \min\{i \in \mathbb{N}:\ t_i \ge 1\} \ge 1$, and $m_0 \ge m_\tau$ is such that $\varphi_i > 0$ for all $i \ge m_0$. If the inequality $\varphi_i \le 0$ holds for arbitrarily large $i \in \mathbb{N}$, we put $\gamma(\beta,\tau) = +\infty$.

Finally, by $\mathbb{T}_0$ we denote the subset of $\mathbb{T}$ that consists of sequences satisfying the condition $\lim\limits_{k\to+\infty} t_k^{-1}t_{k+1} = 1$ of slow growth [15] and the condition $\lim\limits_{k\to+\infty}(t_{k+1}-t_k) = +\infty$.

Theorem 1. *Let $\mathcal{P}$ be a collection of weights. If there exists a set $\Gamma \subset \mathbb{T}_0$ such that the equality $\inf\limits_{\beta\in\mathfrak{N}_1[\mathcal{P}]}\gamma(\beta,\tau) = 0$ holds for any $\tau \in \Gamma$, and for any $p \in \mathcal{P}$ and $M > 0$ there exists a sequence $\tau \in \Gamma$ such that $K_M^\tau \le Cp$ with some $C = C(p,M,\tau) > 0$, then $\mathfrak{N}_n[\mathcal{P}]$ is Γ-ultimate.*

Let $\mathfrak{M}_0[\theta]$ be the set of all perturbations satisfying the estimate $\|Q(t)\| \le N_Q e^{-\sigma\theta(t)}$, where $N_Q \ge 0$, $\sigma > 0$ are numbers depending on Q and $\theta : \mathbb{R}^+ \to]0,+\infty[$ is a fixed piecewise continuous function increasing to $+\infty$ such that $\overline{\lim\limits_{t\to+\infty}}\, t^{-1}\theta(t) < +\infty$. It was proved in [4], [5] that

$$\Lambda(\mathfrak{M}_0[\theta]) = \lim_{\delta\to+0}\Omega(A,\eta(\theta,\delta)), \tag{3}$$

where the sequence $\eta(\theta,\delta) \in \mathbb{T}$ is defined by the recursion formula

$$T_{k+1}(\delta) = T_k(\delta) + \delta\theta(T_k(\delta)),\ \ k \in \mathbb{N}_0, \tag{4}$$

with arbitrary initial condition $T_0(\delta) \ge 0$. The sequence $\eta(\theta,\delta)$ is called the δ-characteristic sequence of θ. This notion was introduced in [4], [5]. It should be stressed that relation (3) is not valid if θ is not monotonic and η is given by (4).

In [14] we define an implicit δ-characteristic sequence of θ by the recurrence relation

$$t_{k+1} = t_k + \delta\theta(t_{k+1}) \tag{5}$$

for continuous non-monotonic functions. It occurs that in general settings of $\theta \in \mathrm{PC}_1(\mathbb{R}^+)$ the appropriate definition can be given in the form

$$\delta\theta(t_{k+1}-0) \ge t_{k+1} - t_k \ge \delta\theta(t_{k+1}+0). \tag{6}$$

Obviously, (6) is equivalent to (5) if θ is continuous. If condition (6) does not define the value of t_{k+1} uniquely , we consider the set S_k of all values satisfying (6) and take the minimal element. It can be proved that the required minimal value exists if S_k is not empty.

We denote the set of all implicit δ-characteristic sequences of θ by $\mathbb{X}(\theta)$. The element of $\mathbb{X}(\theta)$ corresponding to certain values of δ and t_0 is denoted by $\xi(\theta,\delta,t_0)$. It can be easily proved that $\mathbb{X}(\theta) \subset \mathbb{T}_0$ if $\overline{\lim\limits_{t\to+\infty}}\, t^{-1}\theta(t) = 0$ and $\theta(t) \to +\infty$ as $t \to +\infty$.

Definition 2. A collection of weights $\mathcal{P}$ is said to be radical if for any $\varepsilon \in]0,1]$ and $p \in \mathcal{P}$ there exist a weight $p_\varepsilon \in \mathcal{P}$ and a number $R_p(\varepsilon) > 0$ such that $p_\varepsilon < R_p(\varepsilon)p^\varepsilon$.

Definition 3. A function $q \in \mathrm{PC}_1(\mathbb{R}^+)$ is said to be moderately discontinuous if q is strictly positive and there exists a number $c_q > 0$ such that $q(t^*+0) \geq c_q q(t^*-0)$ for any discontinuity point t^* of q.

Theorem 2. *Suppose that $\mathcal{P}$ is radical and each $p \in \mathcal{P}$ is left-continuous, moderately discontinuous, and bounded away from zero by some constant $C_p > 1$. If for any $p \in \mathcal{P}$ the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$ hold, then $\mathfrak{N}_n[\mathcal{P}]$ is $\Gamma_{\mathcal{P}}$-ultimate with $\Gamma_{\mathcal{P}} = \{\xi(\ln p, \delta, t_p) :\ p \in \mathcal{P},$ $;\delta \in]0,1]\} \subset \mathbb{T}_0$, where the mapping $\mathcal{P} \ni p \mapsto t_p \in \mathbb{R}^+$ is arbitrary.*

Corollary 1. *If $\mathcal{P}$ is radical and each $p \in \mathcal{P}$ satisfy the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$, then $\mathfrak{N}_n[\mathcal{P}]$ is Γ-ultimate for some appropriate $\Gamma \subset \mathbb{T}_0$.*

Remark. It can be easily observed from the proof that the inequality

$$\Lambda(\mathfrak{N}_n[\mathcal{P}]) \leq \sup_{\tau \in \Gamma_{\mathcal{P}}} \Omega(A, \tau)$$

follows from the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$, whereas the rest of conditions of Theorem 2 is used only to prove the opposite relation. So we are motivated to consider some radicalization operation on weight collections.

Corollary 2. *Any collection of weights $\mathcal{P}$ such that each $p \in \mathcal{P}$ satisfies the conditions $\lambda[p] = 0$ and $p(t) \to +\infty$ as $t \to +\infty$ may be extended to a colection $\bar{\mathcal{P}}$ such that $\mathfrak{N}_n[\bar{\mathcal{P}}]$ is Γ-ultimate for some appropriate $\Gamma \subset \mathbb{T}_0$.*

Acknowledgement

The work was supported by BRFFR, grant No Φ14P-011.

References

[1] B. F. Bylov, R. È. Vinograd, D. M. Grobman, V. V. Nemyckiĭ, Theory of Ljapunov exponents and its application to problems of stability. (Russian) *Izdat. "Nauka", Moscow*, 1966.

[2] N. A. Izobov, Lyapunov exponents and stability. *Cambridge Scientific Publishers, Cambridge*, 2013.

[3] N. A. Izobov, The highest exponent of a linear system with exponential perturbations. (Russian) *Differencial'nye Uravnenija* **5** (1969), 1186–1192.

[4] E. A. Barabanov, On extreme Lyapunov exponents of linear systems under exponential and power perturbations. (Russian) *Differencial'nye Uravnenija* **20** (1984), No. 2, p. 357.

[5] E. A. Barabanov, Exact bounds of Lyapunov extreme exponents of linear differential systems under exponential and power perturbations. *Cand. Sci. Phys. Math. Dissertation, Minsk*, 1984.

[6] E. A. Barabanov and O. G. Vishnevskaya, Sharp bounds for Lyapunov exponents of a linear differential system with perturbations integrally bounded on the half-line. (Russian) *Dokl. Akad. Nauk Belarusi* **41** (1997), No. 5, 29–34.

[7] E. K. Makarov, I. V. Marchenko, and N. V. Semerikova, On an upper bound for the higher exponent of a linear differential system with perturbations integrable on the half-axis. (Russian) *Differ. Uravn.* **41** (2005), No. 2, 215–224, 286–287; translation in *Differ. Equ.* **41** (2005), No. 2, 227–237.

[8] I. V. Marchenko, A sharp upper bound on the mobility of the highest exponent of a linear system under perturbations that are small in weighted mean. (Russian) *Differ. Uravn.* **41** (2005), No. 10, 1416–1418, 1439; translation in *Differ. Equ.* **41** (2005), No. 10, 1493–1495.

[9] E. K. Makarov and I. V. Marchenko, An algorithm for constructing attainable upper bounds for the highest exponent of perturbed systems. (Russian) *Differ. Uravn.* **41** (2005), No. 12, 1621–1634, 1726; translation in *Differ. Equ.* **41** (2005), No. 12, 1694–1709.

[10] V. M. Millionshchikov, A proof of accessibility of the central exponents of linear systems. (Russian) *Sibirsk. Mat. Zh.* **10** (1969), 99–104.

[11] I. N. Sergeev, Sharp upper bounds of mobility of the Ljapunov exponents of a system of differential equations and the behavior of the exponents under perturbations approaching. (Russian) *Differentsial'nye Uravneniya* **16** (1980), No. 3, 438–448, 572.

[12] I. N. Sergeev, Mobility limits of the Lyapunov exponents of linear systems under small average perturbations. (Russian) *Tr. Semin. Im. I. G. Petrovskogo* **11** (1986), 32–73; translation in *J. Sov. Math.* **45** (1989), No. 5, 1389–1421.

[13] N. A. Izobov, Exponential indices of a linear system and their calculation. (Russian) *Dokl. Akad. Nauk BSSR* **26** (1982), No. 1, 5–8, 92.

[14] E. K. Makarov, On limit classes of bounded perturbations. (Russian) *Differ. Uravn.* **50** (2014), No. 10, 1339–1346; translation in *Differ. Equ.* **50** (2014), No. 10, 1329–1335.

[15] N. A. Izobov, On a set of lower indices of a linear differential system. (Russian) *Differencial'nye Uravnenija* **1** (1965), 469–477.