Asymptotic Representation of Solutions of n-th Order Ordinary Differential Equations with Regularly Varying Nonlinearities

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We consider the differential equation

$$y^{(n)}=\sum_{k=1}^{m}\alpha_k p_k(t)\prod_{j=0}^{n-1}\varphi_{kj}(y^{(j)}), \tag{1}$$

where $\alpha_k\in\{-1;1\}$ $(k=\overline{1,m})$, $p_k:[a,\omega[\,\to\,]0,+\infty[$ $(k=\overline{1,m})$ are continuous functions, $\varphi_{kj}:\Delta Y_j\to\,]0,+\infty[$ $(k=\overline{1,m};\ j=\overline{0,n-1})$ are continuous and regularly varying functions as $y^{(j)}\to Y_j$ of orders σ_{kj}, $-\infty<a<\omega\le+\infty$[1], ΔY_j – one-sided neighborhood of Y_j, Y_j is equal to 0, or $\pm\infty$.

Definition. A solution y of the equation (1) is called a $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solution, where $-\infty\le\lambda_0\le+\infty$, if it is defined on the interval $[t_0,\omega[\,\subset[a,\omega[$ and satisfies the following conditions

$$y^{(j)}(t)\in\Delta_{Y_j}\ \text{ at }\ t\in[t_0,\omega[\,,\quad \lim_{t\uparrow\omega}y^{(j)}(t)=Y_j\ \ (j=\overline{0,n-1}), \tag{2}$$

$$\lim_{t\uparrow\omega}\frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)y^{(n-2)}(t)}=\lambda_0.$$

The aim of this work is to establish the conditions of the existence and asymptotic as $t\to\omega(\omega\le+\infty)$ representations of one class of P_ω solutions of n-th order differential equation (1) containing the right side several main terms, what means that for some $s\in\{1,\dots,m\}$ and not empty set $\Gamma\subset\{1,\dots,m\}$,

$$\lim_{t\uparrow\omega}\frac{p_k(t)\prod\limits_{j=0}^{n-1}\varphi_{kj}(y^{(j)}(t))}{p_s(t)\prod\limits_{j=0}^{n-1}\varphi_{sj}(y^{(j)}(t))}=c_{ks}=const\ne0\ \text{ at }\ k\in\Gamma, \tag{3}$$

$$\lim_{t\uparrow\omega}\frac{p_k(t)\prod\limits_{j=0}^{n-1}\varphi_{kj}(y^{(j)}(t))}{p_s(t)\prod\limits_{j=0}^{n-1}\varphi_{sj}(y^{(j)}(t))}=0\ \text{ at }\ k\in\{1,\dots,m\}\setminus\Gamma. \tag{4}$$

In the works by Evtukhov V. M. and Klopot A. M. [1–3] there is considered the case when in the target class of solutions the right side of equation (1) has one main term, which means that the condition (4) is satisfied for all $k\ne s$.

Let us introduce notation needed in forthcoming considerations.

From the definition of $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solutions of the equation (1) it is clear that any such solution and all of its derivatives up to order n differs from zero on an interval $[t_1,\omega[\,\subset[t_0,\omega[$, and

[1]We consider that $a>1$ when $\omega=+\infty$, and $\omega-1<a<\omega$ when $\omega<+\infty$.

on this interval $j+1$-th ($j \in \{0,\dots,n-1\}$) derivative of this decision is positive, if Δ_{Y_j} is left neighborhood of Y_j, and negative – otherwise. Given this fact enter the number

$$\nu_j = \begin{cases} 1, & \text{if } \Delta_{Y_j}\text{-left neighborhood 0, and if } Y_j = +\infty, \text{ or } Y_j = 0, \\ -1, & \text{if } \Delta_{Y_j}\text{-right neighborhood 0, and if } Y_j = -\infty, \text{ or } Y_j = 0, \end{cases} \quad (j = \overline{0, n-2})$$

defining accordingly signs of j-th and $j+1$-th derivatives of $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solutions. At the same time, we note that for $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$-solutions of equation (1) the conditions

$$\nu_j\nu_{j+1} < 0, \ \text{ if } \ Y_j = 0, \quad \nu_j\nu_{j+1} > 0, \quad Y_j = \pm\infty \ \ (j = \overline{0, n-2}) \tag{5}$$

are satisfied.

Let

$$a_{0i} = (n-i)\lambda_0 - (n-i-1) \ \ (i = 1,\dots,n) \ \text{ at } \ \lambda_0 \in \mathbb{R},$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t-\omega, & \text{if } \ \omega < +\infty. \end{cases}$$

For the formulation of the main results, we introduce the following notation.

$$\gamma_k = 1 - \sum_{j=0}^{n-1} \sigma_{kj}, \quad \mu_{kn} = \sum_{j=0}^{n-2} \sigma_{kj}(n-j-1),$$

$$C_k = \prod_{j=0}^{n-2} \left| \frac{(\lambda_0 - 1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} \right|^{\sigma_{kj}}, \quad J_{kn}(t) = \int_{A_{kn}}^{t} p_k(\tau)|\pi_\omega(\tau)|^{\mu_{kn}}\, d\tau,$$

$$A_{kn} = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p_k(t)|\pi_\omega(t)|^{\mu_{kn}}\, dt = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p_k(t)|\pi_\omega(t)|^{\mu_{kn}}\, dt < +\infty, \end{cases}$$

where $k = \overline{1, m}$,

$$Y(t) = \left| \gamma_s C_s J_{sn}(t) \prod_{j=0}^{n-1} L_{sj}\big(\nu_j |\pi_\omega(t)|^{\frac{a_{0j+1}}{\lambda_0 - 1}}\big) \sum_{k\in\Gamma} \alpha_k c_{ks} \right|,$$

$$Y_j(t) = \nu_{n-1}|Y(t)|^{\frac{1}{\gamma_s}} \frac{[(\lambda_0 - 1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{k=j+1}^{n-1} a_{0k}}.$$

We say that a continuous and slowly varying as $y \to Y$ function $L : \Delta_Y \to]0, +\infty[$ (Y is equal to 0 or $\pm\infty$, Δ_Y – one-sided neighborhood of Y) satisfies the condition S if for any continuously differentiable function $l : \Delta_Y \to]0, +\infty[$ such that

$$\lim_{\substack{y\to Y \\ y\in\Delta_Y}} \frac{y\, l'(y)}{l(y)} = 0,$$

the asymptotic relation holds

$$L(yl(y)) = L(y)[1 + o(1)] \ \text{ for } \ y \to Y \ \ (y \in \Delta_Y).$$

Theorem 1. *Let* $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\}$ *and for some* $s \in \{1, \dots, m\}$ *and not empty set* $\Gamma \subset \{1, \dots, m\}$ *complied inequality* $\gamma_s \neq 0$. *Suppose, moreover, that slowly varying components* $L_{kj}(y)$ $\forall\, k \in \Gamma$ $(j = 0, \dots, n-1)$ *of functions* φ_{kj} *satisfy the condition* S. *Then for the existence of* $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$*-solution of* (1) *for which performed* (3), *where* $\sum\limits_{k\in\Gamma} \alpha_k c_{ks} \neq 0$ *and* (4), *it is necessary the inequalities* (5), *the condition*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J'_{sn}(t)}{J_{sn}(t)} = \frac{\gamma_s}{\lambda_0 - 1}, \tag{6}$$

the inequalities

$$\begin{gathered} \nu_j \nu_{j+1} a_{0j+1} (\lambda_0 - 1) \pi_\omega(t) > 0 \ \ (j = \overline{0, n-2}), \\ \nu_{n-1} \gamma_s J_{sn}(t) \Big(\sum_{k\in\Gamma} \alpha_k c_{ks} \Big) > 0 \ \ \text{at} \ \ t \in\,]a, \omega[\,, \end{gathered} \tag{7}$$

as well as the conditions

$$\lim_{t\uparrow\omega} \frac{p_k(t) \prod\limits_{j=0}^{n-1} \varphi_{kj}(Y_j(t))}{p_s(t) \prod\limits_{j=0}^{n-1} \varphi_{sj}(Y_j(t))} = 0 \ \ \text{at} \ \ k \in \{1, \dots, m\} \setminus \{s\}, \tag{8}$$

$$\lim_{t\uparrow\omega} \frac{p_k(t) \prod\limits_{j=0}^{n-1} \varphi_{kj}(Y_j(t))}{p_s(t) \prod\limits_{j=0}^{n-1} \varphi_{sj}(Y_j(t))} = c_{ks} \ \ \text{at} \ \ k \in \Gamma \tag{9}$$

to be satisfied. Moreover, each such solution as $t \uparrow \omega$ *has the asymptotic representation*

$$y^{(j-1)}(t) = \nu_{n-1} \frac{[(\lambda_0 - 1)\pi_\omega(t)]^{n-j}}{\prod\limits_{i=j}^{n-1} a_{0i}} |Y(t)|^{\frac{1}{\gamma_s}} [1 + o(1)] \ \ (j = \overline{1, \dots, n}), \tag{10}$$

where

$$L_{sj}(y^{(j)}) = |y^{(j)}|^{-\sigma_{sj}} \varphi_{sj}(y^{(j)}) \ \ (j = \overline{0, \dots, n-1}).$$

Let us introduce the following notation.

$$B_m = \frac{\sum\limits_{k\in\Gamma} \alpha_k c_{ks} \sigma_{km}}{\sum\limits_{k\in\Gamma} \alpha_k c_{ks}}\,. \tag{11}$$

Theorem 2. *Let the conditions of Theorem* 1 *be executed. Then, if in addition to* (5), (6), (7), (8) *and* (9) *the algebraic respect to* ρ *equation*

$$\sum_{m=0}^{n-1} B_m \prod_{i=m+1}^{n-1} a_{0i} \prod_{j=1}^{m} (a_{0j} + \rho) = \prod_{j=1}^{n} (a_{0j} + \rho) \tag{12}$$

doesn't have roots with a zero real part, then the differential equation (1) *has* $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$*-solutions of the type* (10). *Moreover, there is an* l*-parameter family of solutions with these representations when among the roots an algebraic equation* (12) *there are* l *roots of real parts which have the opposite sign of* $\beta(\lambda_0 - 1)$.

References

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