The Asymptotic Properties of Rapidly Varying Solutions of Second Order Differential Equations with Regularly and Rapidly Varying Nonlinearities

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The investigations of second order differential equations play an important role in the development of the qualitative theory of differential equations. Such equations have a lot of applications in different fields of science.

Many results have been obtained for equations with power nonlinearities. But in practice we often deal with differential equations not only with power nonlinearities but also with exponential nonlinearities. It happens, for example, when we study the distribution of electrostatic potential in a cylindrical volume of plasma of products of burning. The corresponding equation may be reduced to the following one

$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$

In the work of V. M. Evtukhov and N. G. Drik [3] some results on asymptotic behavior of solutions of such equations have been obtained.

Exponential nonlinearities form a special class of rapidly varying nonlinearities. The consideration of the last ones is necessary for some models. Such consideration needs the establishment of the next class of functions.

The function $\varphi : [s, +\infty[\to]0, +\infty[$ $(s > 0)$ is called a rapidly varying [1] function of the order $+\infty$ as $z \to \infty$ if this function is measurable and the following condition is true

$$\lim_{z\to\infty} \frac{\varphi(\lambda z)}{\varphi(z)} = \begin{cases} 0, & \text{if } 0 < \lambda < 1, \\ 1, & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1. \end{cases}$$

The function φ is called a rapidly varying function of the order $-\infty$ as $z \to \infty$ if this function is measurable and

$$\lim_{z\to\infty} \frac{\varphi(\lambda z)}{\varphi(z)} = \begin{cases} -\infty, & \text{if } 0 < \lambda < 1, \\ 1, & \text{if } \lambda = 1, \\ 0, & \text{if } \lambda > 1. \end{cases}$$

The function $\varphi(z)$ is called a rapidly varying function in zero if $\varphi(\frac{1}{z})$ is a rapidly varying function of the order $+\infty$.

An exponential function is a special case of such functions. The differential equation

$$y'' = \alpha_0 p(t) \varphi(y)$$

with a rapidly varying function φ, was investigated in the work of V. M. Evtuhov and V. M. Kharkov [4]. But in the mentioned work the introduced class of solutions of the equation depends on the function φ. This is not convenient for practice.

The more general class of equations of the mentioned type is established in this work. It is a natural generalization of previous investigations.

Let us consider the differential equation

$$y'' = \alpha_0 p(t) \varphi_0(y) \varphi_1(y'), \tag{1}$$

In this equation α_0 is -1 or $+1$, $p:[a,\omega[\to]0,+\infty[$ $(-\infty<a<\omega\le+\infty)$ is a continuous function, $\varphi_i:\Delta_{Y_i}\to]0,+\infty[$ $(i\in\{0,1\})$ also are continuous functions, $Y_i\in\{0,\pm\infty\}$, the intervals Δ_{Y_i}, $i\in\{0,1\}$ may be of the form $[y_i^0,Y_i[$ [1], or of the form $]Y_i,y_i^0]$.

Furthermore, we assume that the function φ_1 is a regularly varying function as $z\to Y_1$ $(z\in\Delta_{Y_1})$ of the order σ_1, and the function φ_0 is twice continuously differentiable and satisfies the following limit relations

$$\lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}}\varphi_0(z)\in\{0,+\infty\},\qquad \lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}}\frac{\varphi_0(z)\varphi_0''(z)}{(\varphi_0'(z))^2}=1.$$

It can be proved that φ_0 is a rapidly varying function as $z\to Y_0$ $(z\in\Delta_{Y_0})$.

We introduce the following notations and definitions.

$$\pi_\omega(t)=\begin{cases}t, & \text{if } \omega=+\infty,\\ t-\omega, & \text{if } \omega<+\infty,\end{cases}$$

$$\theta_1(z)=\varphi_1(z)|z|^{-\sigma_1},$$

$$I(t)=|\lambda_0-1|^{\frac{1}{1-\sigma_1}}\int\limits_{B_\omega^0}^{t}\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}\operatorname{sign}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega^0=\begin{cases}b, & \text{if } \displaystyle\int\limits_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}\operatorname{sign}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\[2ex] \omega, & \text{if } \displaystyle\int\limits_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}\operatorname{sign}y_1^0\big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\end{cases}$$

$$I_1(t)=\int\limits_{B_\omega^1}^{t}\frac{\lambda_0I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau,\quad B_\omega^1=\begin{cases}b, & \text{if } \displaystyle\int\limits_b^\omega\frac{\lambda_0I(\tau)}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=+\infty,\\[2ex] \omega, & \text{if } \displaystyle\int\limits_b^\omega\frac{\lambda_0|I(\tau)|}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau<+\infty\end{cases}$$

$$\Phi_0(z)=\int\limits_{A_\omega^0}^{z}\frac{1}{|\varphi_0(y)|^{\frac{1}{1-\sigma_1}}}\,dy,\quad A_\omega^0=\begin{cases}y_0^0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}\frac{1}{|\varphi_0(y)|^{\frac{1}{1-\sigma_1}}}\,dy=+\infty,\\[2ex] Y_0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}\frac{1}{|\varphi_0(y)|^{\frac{1}{1-\sigma_1}}}\,dy<+\infty,\end{cases}$$

$$\Phi_1(z)=\int\limits_{A_\omega^1}^{z}\frac{\Phi_0(\tau)}{\tau}\,d\tau,\quad A_\omega^1=\begin{cases}y_0^0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}\frac{\Phi_0(\tau)}{\tau}\,d\tau=+\infty,\\[2ex] Y_0, & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}\frac{|\Phi_0(\tau)|}{\tau}\,d\tau<+\infty.\end{cases}$$

[1] If $Y_i=+\infty$ $(Y_i=-\infty)$, we suppose that $y_i^0>0$ $(y_i^0<0)$.

The inferior limits of the integrals are chosen in such a way that the corresponding integrals tend either to 0 or to ∞.

The solution y of the equation (1) is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution if

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i} \ \ (t_0 \geq a), \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)\, y(t)} = \lambda_0.$$

Let $Y \in \{0, \infty\}$, Δ_Y is some one-sided neighborhood of Y. The differentiable function $L : \Delta_Y \to]0; +\infty[$ is said to be a normalized slowly varying function as $z \to Y$ $(z \in \Delta_Y)$ if

$$\lim_{\substack{z\to Y_1\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0.$$

We say that a slowly varying function $\theta : \Delta_Y \to]0; +\infty[$ as $z \to Y$ $(z \in \Delta_Y)$ satisfies the condition S if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0; +\infty[$ the following relation takes place

$$\theta(zL(z)) = \theta(z)(1 + o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

The next theorem contains necessary and sufficient conditions of existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1), and the asymptotic representations for these solutions and their derivatives of the first order as $t \uparrow \omega$.

Theorem. *Let $\sigma_1 \neq 1$, the function φ_1 satisfy the condition S, and the following limit relation be true*

$$\lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}} \frac{(\frac{\Phi_1'(z)}{\Phi_1(z)})''(\frac{\Phi_1'(z)}{\Phi_1(z)})}{((\frac{\Phi_1'(z)}{\Phi_1(z)})')^2} = \gamma_0, \ \ \gamma_0 \in R \setminus \{1, 0\}.$$

The following conditions are necessary for the existence of the $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1), where $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$,

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0; \quad \pi_\omega(t) y_1^0 \alpha_0(\lambda_0 - 1) > 0, \quad y_1^0 \cdot \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1,$$

$$\lim_{t\uparrow\omega} \Phi_1^{-1}(I_1(t)) = Y_0,$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) I_1'(t)}{I_1(t)} = \infty, \quad \lim_{t\uparrow\omega} \frac{I_1'(t)\pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t))} = \frac{\lambda_0}{\lambda_0 - 1}\,.$$

These conditions are also sufficient for the existence of the $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1) if

$$I(t) I_1(t) \lambda_0(\sigma_1 - 1) > 0 \ \text{ as } \ t \in [a; \omega[$$

and the function $\frac{|\pi_\omega(t)|^{1 - \frac{(2-\gamma_0)\lambda_0}{(1-\gamma_0)(\lambda_0 - 1)}} I_1'(t)}{I_1(t)}$ *is a normalized slowly varying function as $t \uparrow \omega$.*

Moreover, for each such solution the following asymptotic representations take place as $t \uparrow \omega$:

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{y'(t)\Phi_1'(y(t))}{\Phi_1(y(t))} = \frac{I_1'(t)}{I_1(t)}\,[1 + o(1)].$$

References

[1] V. Marić, Regular variation and differential equations. Lecture Notes in Mathematics, 1726. *Springer-Verlag, Berlin*, 2000.

[2] V. M. Evtukhov and A. M. Samoilenko, Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point. *Ukrainian Math. J.* **62** (2010), No. 1, 56–86.

[3] V. M. Evtukhov and N. G. Drik, Asymptotic behavior of solutions of a second-order nonlinear differential equation. *Georgian Math. J.* **3** (1996), No. 2, 101–120.

[4] V. M. Evtukhov and V. M. Khar'kov, Asymptotic representations of solutions of second-order essentially nonlinear differential equations. (Russian) *Differ. Uravn.* **43** (2007), No. 10, 1311–1323, 1437.