Convergence Analysis of Difference Schemes for Generalized Benjamin–Bona–Mahony–Burgers Equation

Givi Berikelashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;
Department of Mathematics, Georgian Technical University, Tbilisi, Georgia
E-mail: bergi@rmi.ge; berikela@yahoo.com

We consider the initial boundary-value problem for the 1D nonlinear Generalized Benjamin–Bona–Mahony–Burgers (GBBM-Burgers) equation

$$\frac{\partial u}{\partial t}-\alpha\,\frac{\partial^2 u}{\partial x^2}+\beta\,\frac{\partial u}{\partial x}+\frac{\partial (u)^m}{\partial x}-\frac{\partial^3 u}{\partial x^2\partial t}=0,\ \ (x,t)\in Q, \tag{1}$$

$$u(0,t)=u(1,t)=0,\ t\in[0,T),\quad u(x,0)=\varphi(x),\ x\in[0,1], \tag{2}$$

where $u(x,t)$ represents the velocity of fluid in the horizontal direction x, $Q=(0,1)\times(0,T]$, $\alpha>0$, β are constants and $m\geq 2$ is an integer.

Assume that the solution of this problem belongs to the fractional-order Sobolev space $W_2^k(Q)$, $k>1$, whose norms we denote by $\|\cdot\|_{W_2^k(Q)}$.

In [1], Che et al. have investigated a three-level unconditionally stable difference scheme for the problem (1), (2) and ascertained second-order convergence under assumption that the exact solution belongs to $\mathcal{C}^{4,3}(\overline{Q})$.

In this article, two-level scheme is constructed to find the values of the unknown function on the first level, besides the term $\partial(u)^2/\partial x$ is approximated by the offered in [2, 3] way. For the upper layers, as in [1], the known approximations are used for derivatives. The error estimate is derived using certain well-known techniques (see, e.g. [4, 5]).

The finite domain $[0,1]\times[0,T]$ in plane is divided into rectangle grids by the points $(x_i,t_j)=(ih,j\tau)$, $i=0,1,\dots,n$, $j=0,1,2,\dots,J$, where $h=1/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

The value of mesh function U at the node (x_i,t_j) is denoted by U_i^j, that is $U(ih,j\tau)=U_i^j$. For the sake of simplicity sometimes we use notations without subscripts: $U_i^j=U$, $U_i^{j+1}=\widehat{U}$, $U_i^{j-1}=\check{U}$. Moreover, let

$$\overline{U}^{\,0}=\frac{U^1+U^0}{2},\quad \overline{U}^{\,j}=\frac{U^{j+1}+U^{j-1}}{2},\ \ j=1,2,\dots\,.$$

We define the difference quotients in x and t directions as follows:

$$(U_i)_{\overline{x}}=\frac{U_i-U_{i-1}}{h}\,,\quad (U_i)_{\overset{\circ}{x}}=\frac{1}{2h}(U_{i+1}-U_{i-1}),\quad (U_i)_{\overline{x}x}=\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}\,,$$
$$(U^j)_t=\frac{U^{j+1}-U^j}{\tau}\,,\quad (U^j)_{\overset{\circ}{t}}=\frac{U^{j+1}-U^{j-1}}{2\tau}\,,\quad (U^j)_{\overline{t}t}=\frac{U^{j+1}-2U^j+U^{j-1}}{\tau^2}\,.$$

We approximate the problem (1), (2) with the help of the three-level finite-difference scheme:

$$\mathcal{L}U_i^j=0,\ \ i=1,2,\dots,n-1,\ \ j=0,1,\dots,J-1, \tag{3}$$

$$U_0^j=U_n^j=0,\ \ j=0,1,2,\dots,J,\quad U_i^0=\varphi(x_i),\ \ i=0,1,2,\dots,n, \tag{4}$$

where

$$
\begin{aligned}
\mathcal{L}U^0 &:= (U^0)_t - \alpha(\overline{U}^0)_{\overline{x}x} + \beta(\overline{U}^0)_{\overset{\circ}{x}} + \frac{m}{m+1}\,\Lambda U^0 - (U^0)_{\overline{x}xt},\\
\mathcal{L}U^j &:= (U^j)_{\overset{\circ}{t}} - \alpha(\overline{U}^j)_{\overline{x}x} + \beta(\overline{U}^j)_{\overset{\circ}{x}} + \frac{m}{m+1}\,\Lambda U^j - (U^j)_{\overline{x}x\overset{\circ}{t}},\ \ j = 1,2,\dots,\\
\Lambda U_i^0 &:= (U_i^0)^{m-1}(\overline{U}^0)_{\overset{\circ}{x}} + \big((U_i^0)^{m-1}\overline{U}^0\big)_{\overset{\circ}{x}},\\
\Lambda U^j &:= (U^j)^{m-1}(\overline{U}^j)_{\overset{\circ}{x}} + \big((U^j)^{m-1}\overline{U}^j\big)_{\overset{\circ}{x}},\ \ j = 1,2,\dots .
\end{aligned}
$$

Let $\overline{\omega} = \{x_i : i = 0,1,2,\dots,n\}$, $\omega = \{x_i : i = 1,2,\dots,n-1\}$, $\omega^+ = \{x_i : i = 1,2,\dots,n\}$. By $L_2(\omega)$ we denote the set of functions defined on the mesh $\bar{\omega}$ and equal to zero at $x = x_0$ and $x = x_n$. We define the following inner product and norms:

$$
(U,V) = \sum_{x\in\omega} hU(x)V(x), \quad \|U\| = (U,U)^{1/2}.
$$

Let, moreover,

$$
(U,V\,] = \sum_{x\in\omega^+} hU(x)V(x), \quad \|U\,]| = (U,U\,]^{1/2}, \quad \|U\|_{W_2^1(\omega)} = \|U_{\overline{x}}]|.
$$

Theorem 1. *Difference scheme* (3), (4) *is uniquely solvable and the following estimates hold for its solution:*

$$
\|U^j\|^2 + \|U^j_{\overline{x}}\,]|^2 \le \|\varphi\|^2 + \|\varphi_{\overline{x}}]|^2, \ \ j = 1,2,\dots .
$$

Theorem 2. *Difference scheme* (3), (4) *is absolutely stable with respect to initial data.*

Theorem 3. *Let the exact solution of the initial-boundary value problem* (1), (2) *belong to* $W_2^k(Q)$. *Then, the convergence rate of the finite difference scheme* (3), (4) *is determined by the estimate*

$$
\|U^j - u^j\|_{W_2^1(\omega)} \le c(\tau^{k-1} + h^{k-1})\|u\|_{W_2^k(Q)}, \ \ 1 < k \le 3,
$$

where $c = c(u)$ *denotes positive constant, independent of* h *and* τ.

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