On the Structure of Upper Frequency Spectra of Linear Differential Equations

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Let us consider the linear nth-order homogeneous differential equation ($n \in \mathbb{N}$)

$$y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_{n-1}(t)\dot{y} + a_n(t)y = 0, \quad t \in \mathbb{R}_+ \stackrel{\text{def}}{=} [0, +\infty), \tag{1}$$

with continuous coefficients $a_i(\,\cdot\,)\colon \mathbb{R}_+ \to \mathbb{R}$, $i = \overline{1, n}$. Identifying equation (1) and its row of coefficients $a = a(\,\cdot\,) = (a_1(\,\cdot\,), \dots, a_n(\,\cdot\,))$, we denote equation (1) also by a. For the set of all nonzero solutions of equation (1) we use the notation $S_*(a)$.

The following definitions were given by I. N. Sergeev [1], [2].

Definition 1. For an arbitrary solution $y(\,\cdot\,) \in S_*(a)$ and a time $t > 0$ the expression $\nu(y, t)$ with either $\nu = \nu^0$ or $\nu = \nu^-$ or $\nu = \nu^+$ is understood as follows.

(a) The number $\nu^0(y, t)$ of zeros of the function $y(\,\cdot\,)$ on the interval $(0, t)$.

(b) The number $\nu^-(y, t)$ of sign alternations of the functions $y(\,\cdot\,)$ on the interval $(0, t)$. (A point $\tau > 0$ is called a sign alternation point of the function $y(\,\cdot\,)$ if in every sufficiently small neighborhood of τ the function takes values of different signs).

(c) The total number $\nu^+(y, t)$ of roots of the function $y(\,\cdot\,)$ on the interval $(0, t)$; here each root of the function $y(\,\cdot\,)$ is counted with regard of their multiplicity.

It is easy to see that $\nu^0(y, t)$, $\nu^-(y, t)$, and $\nu^+(y, t)$ are finite integer numbers for every nonzero solution $y(\,\cdot\,)$ and $t > 0$.

Definition 2. The upper frequencies of zeros, signs, and roots of a solution $y(\,\cdot\,) \in S_*(a)$ are defined as

$$\widehat{\nu}^{\,0}[y] \stackrel{\text{def}}{=} \varlimsup_{t\to+\infty} \frac{\pi}{t}\,\nu^{\,0}(y(\,\cdot\,);t), \quad \widehat{\nu}^{\,-}[y] \stackrel{\text{def}}{=} \varlimsup_{t\to+\infty} \frac{\pi}{t}\,\nu^{\,-}(y(\,\cdot\,);t), \ \text{ and } \ \widehat{\nu}^{\,+}[y] \stackrel{\text{def}}{=} \varlimsup_{t\to+\infty} \frac{\pi}{t}\,\nu^{\,+}(y(\,\cdot\,);t),$$

respectively.

Definition 3. The upper frequency spectra $\widehat{\nu}^{\,0}(S_*(a)$, $\widehat{\nu}^{\,-}(S_*(a)$, and $\widehat{\nu}^{\,+}(S_*(a)$ of zeros, signs, and roots of equation (1) are defined as the sets of upper frequencies of zeros, signs, and roots of all solutions belonging to $S_*(a)$, respectively.

Generally speaking, upper frequencies (2) can be equal to $+\infty$ for some solutions of equation (1) with unbounded coefficients.

For symbols $\nu = \nu^0$, ν^-, and ν^+, respectively, functions $\widehat{\nu}(\,\cdot\,)\colon \mathbb{R}^n \setminus \{\mathbf{0}\} \to \overline{\mathbb{R}}_+$ are defined as $\widehat{\nu}(\alpha) \stackrel{\text{def}}{=} \widehat{\nu}[y_\alpha]$, where $y_\alpha(\,\cdot\,)$ is a solution of equation (1) such that $(y_\alpha(0), \dot{y}_\alpha(0), \dots, y_\alpha^{(n-1)}(0))^{\mathrm{T}} = \alpha$, and $\overline{\mathbb{R}}_+ \stackrel{\text{def}}{=} [0, +\infty]$ is a nonnegative semi-axis of the extended real number line $\overline{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \sqcup \{-\infty, +\infty\}$. The functions $\widehat{\nu}^{\,0}(\,\cdot\,)$, $\widehat{\nu}^{\,-}(\,\cdot\,)$, and $\widehat{\nu}^{\,+}(\,\cdot\,)$ are called functions of zeros, signs, and roots of equation (1), respectively.

As it follows from Sturm's theorem and was noted in [1], [2], the upper frequency spectra consist of zero for an arbitrary first-order equation (1) and of the same nonnegative number for an arbitrary second-order equation (1). Let us present some results dedicated to the structure of the upper frequency spectra of higher order equations. For arbitrary positive incommensurable numbers $\omega_2 > \omega_1$ there exists [3] a fourth-order autonomous equation, whose upper frequency spectra coincide with segment $[\omega_1, \omega_2]$. There exists [4] a third-order periodic equation whose upper frequency spectra contain the same segment. In [5] a third-order equation was constructed whose upper frequency spectra are equal to $[0,1] \cap \mathbb{Q}$, where the symbol $\mathbb{Q}$ stands for the set of rational numbers. Moreover in the paper [5] another one third-order equation was obtained whose upper frequency spectra consist of $([0,1] \cap \mathbb{I}) \cup \{0\}$, where by $\mathbb{I}$ we denote the set of irrational numbers of the real number line $\mathbb{R}$.

We naturally encounter the problem as to what the upper frequency spectra and the functions of zeros, signs, and root are. In the report under the assumption that zero belongs to the upper frequency spectra of equation (1) the complete description of the spectra are obtained. Here we also give an improvable description of the functions $\widehat{\nu}^0(\,\cdot\,)$, $\widehat{\nu}^-(\,\cdot\,)$, and $\widehat{\nu}^+(\,\cdot\,)$ in terms of Baire classes.

To formulate the theorem of our report let us briefly give some necessary notations and definitions. Let $\mathcal{M}$ be an arbitrary set and N be some class of its subsets. It is said that a function $f(\,\cdot\,)\colon \mathcal{M} \to \overline{\mathbb{R}}$ belongs to the class $({}^*, N)$ if for every $r \in \overline{\mathbb{R}}$ Lebesgue set $[f(\,\cdot\,) \geqslant r]$ (i.e. a preimage $f^{-1}([r, +\infty])$ of the segment $[r, +\infty]$) belongs to the class N. In the report we consider mainly Borel subsets of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ of orders zero, one, and two [6]. Closed and open sets are said to be Borel sets of zero order. Borel sets of order one are sets of type F_σ or G_δ which are, respectively, countable unions of closed sets and countable intersections of open sets. Borel sets of the second order are set of type $F_{\sigma\delta}$ (the countable intersections of F_σ-sets) or sets of type $G_{\delta\sigma}$ (the countable unions of G_δ-sets). Borel sets of an arbitrary finite order are defined in a similar manner by induction. A set is said to be a Borel set of the exact order k if it is a Borel set of the kth order but it isn't Borel set of order $k-1$.

A set $\mathcal{A} \subset \mathbb{R}$ is called a Suslin set [7, p. 213], [8, p. 489] of the number line $\mathbb{R}$ if it is a continuous image of irrational numbers $\mathbb{I}$ with the subspace topology. The class of Suslin sets contains the class of Borel sets as a proper subclass, and at the same time it is a proper subclass of the class of Lebesgue measurable sets. A set $\mathcal{A} \subset \overline{\mathbb{R}}$ is called a Suslin set of the extended real number line if it can be represented as an union of a Suslin set of $\mathbb{R}$ and some subset (including the empty subset) of two-element set $\{-\infty, +\infty\}$.

Theorem 1. *The following inclusions $\widehat{\nu}^-(\,\cdot\,) \in ({}^*, G_\delta)$ and $\widehat{\nu}^0(\,\cdot\,)$, $\widehat{\nu}^+(\,\cdot\,) \in ({}^*, F_{\sigma\delta})$ hold.*

From Theorem 1 it follows that the function $\widehat{\nu}^-(\,\cdot\,)$ belongs to the second Baire class and the functions $\widehat{\nu}^0(\,\cdot\,)$, $\widehat{\nu}^+(\,\cdot\,)$ belong to the third Baire class. The following theorem is a simple consequence of Theorem 1 and the definition of Suslin sets.

Theorem 2. *The upper frequency spectra $\widehat{\nu}^0(S_*(a))$, $\widehat{\nu}^-(S_*(a))$, and $\widehat{\nu}^+(S_*(a))$ of zeros, signs, and roots of equation* (1) *are Suslin sets of the nonnegative semi-axis $\overline{\mathbb{R}}_+$.*

Under the assumption that zero belongs to the upper frequency spectra the converse of Theorem 2 was obtained.

Theorem 3. *For an arbitrary Suslin set $\mathcal{A} \subset \overline{\mathbb{R}}_+$ containing zero there exists a third-order differential equation* (1) *whose upper frequency spectra of zeros, signs, and roots are equal to $\mathcal{A}$.*

The following theorem shows that the assertion of Theorem 1 is improvable.

Theorem 4. *There exist a number $r > 0$ and a third-order differential equation* (1) *such that the Lebesgue set $[\widehat{\nu}^-(\,\cdot\,) \geqslant r]$ of its function of signs is a Baire set of the exact first order, also there exists another third-order differential equation* (1) *such that the Lebesgue sets $[\widehat{\nu}^0(\,\cdot\,) \geqslant r]$ and $[\widehat{\nu}^+(\,\cdot\,) \geqslant r]$ of its functions of zeros and roots, respectively, are Baire sets of the exact second order.*

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