On the Well-Possedness of the Cauchy Problem and the Lyapunov Stability for Systems of Generalized Ordinary Differential Equations

Malkhaz Ashordia

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;
Sokhumi State University, Tbilisi, Georgia
E-mail: ashord@rmi.ge

Let $A_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$ and $t_0 \in I$, where $I \subset \mathbb{R}$ is an arbitrary interval non-degenerated in the point. Consider the Cauchy problem

$$dx(t) = dA_0(t)\cdot x(t) + df_0(t), \quad x(t_0) = c_0. \tag{1}$$

Let x_0 be the unique solution of problem (1).

Along with the Cauchy problem (1) consider the sequence of the Cauchy problems

$$dx(t) = dA_k(t)\cdot x(t) + df_k(t), \quad x(t_k) = c_k \ \ (k = 1,2,\dots), \tag{1_k}$$

where $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $f_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$, $t_k \in I$ $(k = 1,2,\dots)$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has been stimulated also by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from the unified viewpoint.

In [2–4] the sufficient conditions are given for problem (1_k) to have a unique solution x_k for sufficiently large k and

$$\lim_{k\to+\infty} \|x_k - x_0\|_s = 0. \tag{2}$$

In the present paper, the necessary and sufficient conditions are established for the sequence of the Cauchy problems (1_k) $(k = 1,2,\dots)$ to have the above-mentioned property. Obtained here results are based on the concept given in [8] and they differ from the analogous ones given in [3].

Moreover, we consider the question of relationship between the Lyapunov stability of system given in (1) and the well-possedness of the Cauchy problem (1). Presented below results are more general than analogous ones obtained in [4].

The following notations and definitions will be used.

$\mathbb{R} =]-\infty,+\infty[\,$, $\mathbb{R}_+ = [0,+\infty[\,$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$O_{n\times m}$ is the zero $n\times m$ matrix.

I_n is an identity $n\times n$ matrix.

$\bigvee\limits_a^b(X)$ is the sum of total variations of the components x_{ij} $(i = 1,\dots,m;\ j = 1,\dots,m)$ of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$; $\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point $t \in I$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}(I;\mathbb{R}^{n\times m})$ is the space of all bounded variation matrix-functions $X : I \to \mathbb{R}^{n\times m}$ with the norm $\|X\|_s = \sup\{\|X(t)\| :\ t \in I\}$.

$\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ for which the restriction on $[a,b]$ belong to $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ for every closed interval $[a,b] \subset I$.

$\widetilde{C}_{loc}(I;\mathbb{R}^n)$ is the set of all vector-functions $x : I \to \mathbb{R}^n$ which are absolutely continuous on every closed interval $[a,b]$ from I.

$L(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ whose components are Lebesgue-integrable;

$L_{loc}(I;\mathbb{R}^{n\times m})$ is the set of matrix-functions $X : I \to \mathbb{R}^{n\times m}$ whose components are Lebesgue integrable on every closed interval from I.

We introduce the operators. If $X \in \mathrm{BV}_{loc}(I,\mathbb{R}^{l\times n})$ and $Y : I \to \mathbb{R}^{n\times m}$, then we put

$$\mathcal{B}(X,Y)(t) \equiv X(t)Y(t) - X(t_0)Y(t_0) - \int\limits_{t_0}^{t} dX(\tau)\cdot Y(\tau),$$

$$\mathcal{I}(X,Y)(t) \equiv \int\limits_{t_0}^{t} d(X(\tau) + \mathcal{B}(X,Y)(\tau))\cdot X^{-1}(\tau).$$

If $X \in \mathrm{BV}(I\,;\mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_j X(t)) \neq 0$ for $t \in I$ $(j=1,2)$, and $Y \in \mathrm{BV}([a,b]\,;\mathbb{R}^{n\times m})$, then $\mathcal{A}(X,Y)(t_0) \equiv O_{n\times m}$,

$$\mathcal{A}(X,Y)(t) \equiv Y(t) - Y(t_0) + \\ + \sum_{t_0<\tau\le t} d_1X(\tau)\cdot (I_n - d_1X(\tau))^{-1}\, d_1Y(\tau) - \sum_{t_0\le\tau<t} d_2X(\tau)\cdot (I_n + d_2X(\tau))^{-1}\, d_2Y(\tau).$$

A vector-function $x \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ is said to be a solution of the generalized differential system given in (1) if

$$x(t) - x(s) = \int\limits_{s}^{t} dA_0(\tau)\cdot x(\tau) + f_0(t) - f_0(s) \ \text{ for } \ s<t; \ \ s,t \in I,$$

where integral is understand in the Kurzweil sense [9].

Without loss of generality, we assume that either $t_k < t_0$ $(k=1,2,\dots)$ or $t_k = t_0$ $(k=1,2,\dots)$ or $t_k > t_0$ $(k=1,2,\dots)$.

Definition 1. We say that the sequence $(A_k, f_k; t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(A_0, f_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k=1,2,\dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{3}$$

problem (1_k) has a unique solution x_k for any sufficient large k and condition (2) holds.

Theorem 1. *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$ $(k=0,1,\dots)$, $t_0 \in I$ and the sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that*

$$\begin{gathered} \det\big(I_n + (-1)^j d_j A_0(t)\big) \neq 0 \ \textit{ for } \ t \in I, \quad (-1)^j (t - t_0) < 0 \ \textit{ and for} \\ t = t_0 \ \textit{ if } \ j \in \{1,2\} \ \textit{ is such that } \ (-1)^j (t_k - t_0) > 0 \ \ (k=1,2,\dots), \end{gathered} \tag{4}$$

$$\lim_{k\to+\infty} t_k = t_0. \tag{5}$$

Then

$$\big((A_k, f_k; t_k)\big)_{k=1}^{+\infty} \in \mathcal{S}(A_0, f_0; t_0) \tag{6}$$

if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that*

$$\inf\big\{|\det(H_0(t))|:\ t\in I\big\}>0, \tag{7}$$

$$\lim_{k\to+\infty} H_k(t_k)=H_0(t_0), \tag{8}$$

$$\lim_{k\to+\infty} \|H_k-H_0\|_s=0, \tag{9}$$

$$\lim_{k\to+\infty}\sup_{t\in I}\Big\{\big\|(\mathcal{I}(H_k,A_k)(t)-\mathcal{I}(H_0,A_0)(t))\big\|\big(1+\big|\bigvee_{t_0}^{t}(\mathcal{I}(H_k,A_k))\big|\big)\Big\}=0 \tag{10}$$

and

$$\lim_{k\to+\infty}\sup_{t\in I}\Big\{\big\|(\mathcal{I}(H_k,f_k)(t)-\mathcal{I}(H_0,f_0)(t)\big\|\big(1+\big|\bigvee_{t_0}^{t}(\mathcal{I}(H_k,A_k))\big|\big)\Big\}=0. \tag{11}$$

Definition 2. The Cauchy problem (1) is called well-possed if condition (6) holds for every sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ and t_k $(k=1,2,\dots)$ for which there exists a sequence H_k $(k=0,1,\dots)$ such that conditions (4), (5) and (7)–(11) hold.

The statements of Theorem 1 mean that the Cauchy problem (1) is well-possed.

Definition 3. The Cauchy problem (1) is called weakly well-possed if condition (6) holds for every sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ and t_k $(k=1,2,\dots)$ for which there exists a sequence H_k $(k=0,1,\dots)$ such that conditions (4), (5), (7)–(9) and

$$\lim_{k\to+\infty}\Big(\big\|\mathcal{I}(H_k,A_k)-\mathcal{I}(H_0,A_0)\big\|_s+\big\|\mathcal{I}(H_k,f_k)-\mathcal{I}(H_0,f_0)\big\|_s\Big)=0$$

hold.

Consider now the Lyapunov stability question on the set $I=[0,+\infty[$.

Definition 4. A solution x_0 of the system given in (1) is called uniformly stable if for every $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ such that an arbitrary solution x of system (1), satisfying the inequality

$$\|x(t_0)-x_0(t_0)\|<\delta \tag{12}$$

for some $t_0\in\mathbb{R}_+$, admits the estimate $\|x(t)-x_0(t)\|<\delta$ for $t\ge t_0$.

Definition 5. Let $\xi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing function such that $\lim\limits_{t\to+\infty}\xi(t)=+\infty$. A solution x_0 of the system given in (1) is called ξ-exponentially asymptotically stable if there exists a positive number η such that for every $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ such that an arbitrary solution x of system (1), satisfying inequality (12) for some $t_0\in\mathbb{R}_+$, admits the estimate

$$\|x(t)-x_0(t)\|<\varepsilon\exp\big(-\eta(\xi(t)-\xi(t_0))\big)\ \text{ for }\ t\ge t_0.$$

Note that the exponentially asymptotic stability (see [3]) is a particular case of the ξ-exponentially asymptotic stability if we assume $\xi(t)\equiv t$.

Definition 6. The system given in (1) is called stable in one or another sense if every its solution is stable in the same sense.

Definition 7. The matrix-function A_0 is called stable in one or another sense if the system $dx(t)=dA_0(t)\cdot x(t)$ is stable in the same sense.

Theorem 2. *Let* $A_0\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $f_0\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n})$ *be such that*

$$\limsup_{t\to+\infty}\bigvee_{t}^{\nu(\xi)(t)}\mathcal{A}(A_0,A_0)<+\infty\ \text{ and }\ \lim_{t\to+\infty}\bigvee_{t}^{\nu(\xi)(t)}\mathcal{A}(A_0,f_0)=0,$$

where $\nu(\xi)(t)=\sup\{\tau\ge t:\ \xi(\tau)\le\xi(t)+1\}$. *Then ξ-exponentially asymptotically stability of A_0 guarantees the well-possedness of problem (1) on* $\mathbb{R}_+$.

Theorem 3. *Let $A_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $f_0 \in \mathrm{BV}(\mathbb{R}_+;\mathbb{R}^n)$. Then uniformly stability of A_0 guarantees the weakly well-possedness of problem* (1) *on $\mathbb{R}_+$.*

We realize the above-given results for the Cauchy problem for ordinary differential systems. Given here results are more general than obtained in [1, 5–8].

Let $\mathcal{P}_0 \in L_{loc}(I,\mathbb{R}^{n\times n})$ and $q_0 \in L_{loc}(I,\mathbb{R}^n)$. Let $x_0 \in \widetilde{C}_{loc}(I;\mathbb{R}^n)$ be the unique solution of the Cauchy problem

$$\frac{dx}{dt} = \mathcal{P}_0(t)x + q_0(t), \quad x(t_0) = c_0. \tag{13}$$

Consider the sequence of the Cauchy problems

$$\frac{dx}{dt} = \mathcal{P}_k(t)x + q_k(t), \quad x(t_k) = c_k \ \ (k = 1,2,\dots). \tag{13_k}$$

The system (13_k) is the particular case of system (1_k) if we assume that $A_k(t) \equiv \int\limits_{t_0}^{t} \mathcal{P}_k(\tau)\,d\tau$ and $f_k(t) \equiv \int\limits_{t_0}^{t} q_k(\tau)\,d\tau$ for every $k \in \{0,1,\dots\}$. Therefore, the results given below immediately follow from the analogous ones presented above.

Definition 8. We say that the sequence $(\mathcal{P}_k, q_k, t_k)$ $(k = 1,2,\dots)$ belongs to the set $\mathcal{S}(\mathcal{P}_0, q_0, t_0)$ if condition (2) holds for every $c_0 \in \mathbb{R}^n$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ satisfying the condition (3), where x_k is the unique solution problem (13_k).

Theorem 4. *Let $P_k \in L(I,\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$, $q_k \in L(I;\mathbb{R}^n)$ $(k = 0,1,\dots)$, and the sequence of points $t_k \in I$ $(k = 1,2,\dots)$ satisfy condition* (5)*. Then*

$$\big((\mathcal{P}_k, q_k, t_k)\big)_{k=1}^{+\infty} \in S(\mathcal{P}_0, q_0, t_0) \tag{14}$$

if and only if there exists a sequence of matrix-functions $H_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$ such that conditions (7)–(9),

$$\lim_{k\to+\infty} \sup_{t\in I} \left\{ \left\| \int\limits_{t_0}^{t} (\mathcal{P}_k^*(\tau) - \mathcal{P}_0^*(\tau))\,d\tau \right\| \left(1 + \left| \int\limits_{t_0}^{t} \|\mathcal{P}_k^*(\tau)\|\,d\tau \right| \right) \right\} = 0 \tag{15}$$

and

$$\lim_{k\to+\infty} \sup_{t\in I} \left\{ \left\| \int\limits_{t_0}^{t} (q_k^*(\tau) - q_0^*(\tau))\,d\tau \right\| \left(1 + \left| \int\limits_{t_0}^{t} \|\mathcal{P}_k^*(\tau)\|\,d\tau \right| \right) \right\} = 0 \tag{16}$$

hold, where

$$\mathrm{P}_k^*(t) \equiv \big(H_k'(t) + H_k(\tau)P_k(t)\big)H_k^{-1}(t), \quad q_k^*(t) \equiv \big(H_k'(t) + H_k(\tau)q_k(t)\big)H_k^{-1}(t).$$

Definition 9. The Cauchy problem (13) is called well-possed if condition (14) holds for every sequence $(\mathcal{P}_k, q_k, t_k)$ $(k = 1,2,\dots)$ and t_k $(k = 1,2,\dots)$ for which there exists a sequence H_k $(k = 0,1,\dots)$ such that conditions (7)–(9), (15) and (16) hold, where $\mathcal{P}_k^*$ and q_k^* are matrix- and vector-functions defined in Theorem 4.

Definition 10. The Cauchy problem (1) is called weakly well-possed if condition (14) holds for every sequences $(\mathcal{P}_k, q_k, t_k)$ $(k = 1,2,\dots)$ and t_k $(k = 1,2,\dots)$ for which there exists a sequence H_k $(k = 0,1,\dots)$ such that conditions (7)–(9) and

$$\lim_{k\to+\infty} \sup_{t\in I} \left\{ \left\| \int\limits_{t_0}^{t} \big(\mathcal{P}_k^*(\tau) - \mathcal{P}_0^*(\tau)\big)\,d\tau \right\| + \left\| \int\limits_{t_0}^{t} \big(q_k^*(\tau) - q_0^*(\tau)\big)\,d\tau \right\| \right\} = 0$$

hold, where $\mathcal{P}_k^*$ and q_k^* are the matrix- and vector-functions defined in Theorem 4.

Theorem 5. *Let $\mathcal{P}_0 \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $q_0 \in L_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ be such that*

$$\lim_{t\to+\infty} \sup \int\limits_t^{\nu(\xi)(t)} \|\mathcal{P}_0(\tau)\|\, d\tau < +\infty \ \ \text{and} \ \ \lim_{t\to+\infty} \int\limits_t^{\nu(\xi)(t)} \|q_0(\tau)\|\, d\tau = 0,$$

where $\nu(\xi)(t) = \sup\{\tau \geq t : \ \xi(\tau) \leq \xi(t) + 1\}$. Then ξ-exponentially asymptotically stability of $\mathcal{P}_0$ guarantees the well-possedness of problem (13) *on $\mathbb{R}_+$.*

Theorem 6. *Let $\mathcal{P}_0 \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $q_0 \in L(\mathbb{R}_+, \mathbb{R}^n)$. Then uniformly stability of $\mathcal{P}_0$ guarantees the weakly well-possedness of problem* (13) *on $\mathbb{R}_+$.*

References

[1] M. Ashordia, On the stability of solutions of linear boundary value problems for the system of ordinary differential equations. This paper also appears in *Georgian Math. J.* **1** (1994), No. 2, 115-126. *Proc. Georgian Acad. Sci. Math.* **1** (1993), No. 2, 129–141.

[2] M. Ashordia, On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. This paper also appears in *Georgian Math. J.* **1** (1994), No. 4, 343-351. *Proc. Georgian Acad. Sci. Math.* **1** (1993), No. 4, 385–394.

[3] M. Ashordia, Criteria of correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Czechoslovak Math. J.* **46(121)** (1996), No. 3, 385–404.

[4] M. T. Ashordia and N. A. Kekelia, On the well-posedness of the Cauchy problem for linear systems of generalized ordinary differential equations on an infinite interval. (Russian) *Differ. Uravn.* **40** (2004), No. 4, 443–454, 574; translation in *Differ. Equ.* **40** (2004), No. 4, 477–490.

[5] I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. Linear theory. (Russian) *"Metsniereba", Tbilisi*, 1997.

[6] M. A. Krasnosel'skiĭ and S. G. Krín, On the principle of averaging in nonlinear mechanics. (Russian) *Uspehi Mat. Nauk (N.S.)* **10** (1955), No. 3(65), 147–152.

[7] Y. Kurzweil and Z. Vorel, Continuous dependence of solutions of differential equations on a parameter. (Russian) *Czechoslovak Math. J.* **7 (82)** (1957), 568–583.

[8] Z. Opial, Continuous parameter dependence in linear systems of differential equations. *J. Differential Equations* **3** (1967), 571–579.

[9] Š Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London*, 1979.