Sensitivity Analysis of One Class of Controlled Functional Differential Equation Considering Variable Delay Perturbation and the Continuous Initial Condition

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Sensitivity analysis of the differential equation consists in finding an analytic relation between solutions of the original and perturbed equations. It is an important tool for assessing properties of the mathematical models. For example, in an immune model [1], it allows one to determine dependence of viruses concentrations on the model parameters. In the present work linear representation of the first order sensitivity coefficient is obtained with respect to perturbations of the initial data.

Let $I=[a,b]$ be a finite interval and let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T is the sign of transposition. Suppose that $O\subset\mathbb{R}^n_x$ and $U_0\subset\mathbb{R}^r_u$ are open sets. Let the n-dimensional function $f(t,x,y,u,v)$ satisfies the following conditions: for almost all $t\in I$, the function $f(t,\cdot\,):O^2\times U_0^2\to\mathbb{R}^n_x$ is continuously differentiable; for any $(x,y,u,v)\in O^2\times U_0^2$, the functions $f(t,x,y,u,v)$, $f_x(\cdot\,)$, $f_y(\cdot\,)$, $f_u(\cdot\,)$, $f_v(\cdot\,)$ are measurable on I; for arbitrary compacts $K\subset O$, $U\subset U_0$ there exists a function $m_{K,U}(\cdot\,)\in L(I,[0,\infty))$, such that for any $(x,y,u,v)\in K^2\times U^2$ and for almost all $t\in I$ the following inequality is fulfilled

$$|f(t,x,y,u)|+|f_x(\cdot\,)|+|f_y(\cdot\,)|+|f_u(\cdot\,)|+|f_v(\cdot\,)|\le m_{K,U}(t).$$

Further, let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t)$, $t\in I$, satisfying the conditions:

$$\tau(t)<t,\quad \dot\tau(t)>0,\quad \inf\big\{\tau(a):\ \tau\in D\big\}:=\widehat\tau>-\infty,\quad \sup\{\tau^{-1}(b):\tau\in D\}<\infty,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$.

Let Φ be the set of continuous initial functions $\varphi(t)\in O$, $t\in I_1=[\widehat\tau,b]$ and let Ω be the set of measurable bounded control functions $u(t)\in U_0$, $t\in I_1$, with $u(I_1)\subset U_0$.

To each element (initial date) $\mu=(t_0,\tau,\varphi,u)\in\Lambda=[a,b)\times D\times\Phi\times\Omega$ we assign the controlled delay functional differential equation

$$\dot x(t)=f\big(t,x(t),x(\tau(t)),u(t),u(\theta(t))\big) \tag{1}$$

with the continuous initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0], \tag{2}$$

where $\theta\in D$ is a fixed delay function. The condition (2) is said to be continuous initial condition since always $x(t_0)=\varphi(t_0)$.

Definition 1. Let $\mu=(t_0,\tau,\varphi,u)\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in[\widehat\tau,t_1]$, $t_1\in(t_0,b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\widehat\tau,t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ be a given element and $x_0(t)$ be the solution corresponding to μ_0 and defined on $[\widehat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$. Let us introduce the set of variations

$$V=\Big\{\delta\mu=(\delta t_0,\delta\tau,\delta\varphi,\delta u):\ |\delta t_0|\leq\alpha,\ \|\delta\tau\|\leq\alpha,\ \delta\varphi=\sum_{i=1}^{k}\lambda_i\delta\varphi_i,\ |\lambda_i|\leq\alpha,\ i=\overline{1,k},\ \|\delta u\|\leq\alpha\Big\}.$$

Here $\delta t_0 \in I - t_{00}$, $\delta\tau \in D - \tau_0$, $\|\delta\tau\| = \sup\{|\delta\tau(t)| : t \in I\}$, $\delta u \in \Omega - u_0$, $\delta\varphi_i \in \Phi - \varphi_0$, $i = \overline{1,k}$, are fixed functions, $\alpha > 0$ is a fixed number.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda$ and there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\delta x(t;\varepsilon\delta\mu) = x(t;\mu_0+\varepsilon\delta\mu) - x_0(t),\ \ \forall\,(t,\varepsilon,\delta\mu) \in [\widehat{\tau}, t_{10}+\delta_1]\times(0,\varepsilon_1]\times V.$$

Theorem 1. *Let the following conditions hold:*

1) *the function $\varphi_0(t)$, $t \in I_1$, is absolutely continuous and the function $\dot\varphi_0(t)$ is bounded;*
2) *the function $f_0(z)$, $z = (t, x, y) \in I \times O^2$, is bounded, where $f_0(t, x, y) = f(t, x, y, u_0(t), u_0(\theta(t)))$;*
3) *there exist the finite limits*

$$\dot\varphi_0^- = \dot\varphi_0(t_{00}-),\quad \lim_{z\to z_0} f_0(z) = f_0^-,\ \ z \in (a, t_{00}] \times O^2,$$

where $z_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$.

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that

$$\delta x(t;\varepsilon\delta\mu) = \varepsilon\delta x(t;\delta\mu) + o(t;\varepsilon\delta\mu) \tag{3}$$

for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V^-$, where $V^- = \{\delta\mu \in V : \delta t_0 \leq 0\}$, and

$$\delta x(t;\delta\mu) = Y(t_{00};t)[\dot\varphi_0^- - f_0^-]\delta t_0 + \beta(t;\delta\mu), \tag{4}$$

$$\beta(t;\delta\mu) = Y(t_{00};t)\delta\varphi(t_{00}) + \int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t) f_{0y}[\gamma_0(s)]\dot\gamma_0(s)\delta\varphi(s)\,ds +$$

$$+\int\limits_{t_{00}}^{t} Y(s;t) f_{0y}[s]\dot x_0(\tau_0(s))\delta\tau(s)\,ds + \int\limits_{t_{00}}^{t} Y(s;t)\Big[f_{0u}[s]\delta u(s) + f_{0v}[s]\delta u(\theta(s))\Big]\,ds, \tag{5}$$

$$\lim_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon} = 0 \ \ \text{uniformly for}\ \ (t,\delta\mu) \in [t_{00}, t_{10}+\delta_2]\times V^-, \tag{6}$$

$Y(s;t)$ is $n \times n$-matrix function satisfying the system

$$Y_s(s;t) = -Y(s;t)f_{0x}[s] - Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot\gamma_0(s),\ \ s \in [t_{00}, t],$$

and the condition

$$Y(s;t) = \begin{cases} H, & s = t, \\ \Theta, & s > t, \end{cases}$$

$f_{0x}[s] = f_{0x}(s, x_0(s), x_0(\tau_0(s)))$, $\gamma_0(s)$ is the inverse function of $\tau_0(s)$; H is the identity matrix and Θ is the zero matrix.

Some comments

The function $\delta x(t;\delta\mu)$ is called the first order sensitivity coefficient and the expression (4) is linear representation of sensitivity coefficient. On the other hand, the function $\delta x(t;\delta\mu)$ is called the first variation of solution $x_0(t)$, $t\in[t_{00},t_{10}+\delta_2]$, and the expression (4) is called the variation formula. The variation formulas play an important role in proving the necessary optimality conditions [2–4]. The questions connected with the variation formulas and the sensitivity analysis for the various classes of differential equations are considered in [2–5].

The addend

$$\int\limits_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)\,ds$$

in formula (5) is the effect of perturbation of the delay function $\tau_0(t)$.

The expression

$$Y(t_{00};t)[\dot{\varphi}_0^- - f_0^-]\delta t_0$$

is the effect of continuous initial condition (2) and perturbation of the initial moment t_{00}.

The expression

$$Y(t_{00};t)\delta\varphi(t_{00})+\int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(s);t)f_{0y}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)\,ds$$

in formula (5) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression

$$\int\limits_{t_{00}}^{t} Y(s;t)\Big[f_{0u}[s]\delta u(s)+f_{0v}[s]\delta u(\theta(s))\Big]\,ds$$

in formula (5) is the effect of perturbation of the control function $u_0(t)$.

Theorem 2. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous and let the functions $\dot{\varphi}_0(t)$ and $f_0(z)$, $z\in I\times O^2$, be bounded. Moreover, there exist the finite limits*

$$\dot{\varphi}_0^+=\dot{\varphi}_0(t_{00}+),\quad \lim_{z\to z_0} f_0(z)=f_0^+,\ \ z\in[t_{00},b)\times O^2.$$

Then for each $\widehat{t}_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, where $V^+=\{\delta\mu\in V:\ \delta t_0\geq 0\}$, formula (3) holds, where

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot{\varphi}_0^+ - f_0^+)\delta t_0+\beta(t;\delta\mu).$$

Theorem 3. *Let the assumptions of Theorems 1 and 2 be fulfilled. Moreover,*

$$\dot{\varphi}_0^- = \dot{\varphi}_0^+ := \widehat{\varphi}_0,\ \ f_0^- = f_0^+ := \widehat{f}_0.$$

Then for each $\widehat{t}_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2]\times V$ formula (3) holds, where

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\widehat{\varphi}_0-\widehat{f}_0)\delta t_0+\beta(t;\delta\mu).$$

All assumptions of Theorem 3 are satisfied if: the functions $\dot{\varphi}_0(t)$, $u_0(t)$, $u_0(\theta(t))$ are continuous at the point t_{00} and the function $f(t,x,y,u,v)$ is continuous and bounded. Clearly, in this case $\widehat{\varphi}_0=\dot{\varphi}_0(t_{00})$ and $\widehat{f}_0=f(t_{00},\varphi_0(t_{00}),\varphi_0(\tau_0(t_{00})),u_0(t_{00}),u_0(\theta(t_{00})))$.

Acknowledgement

The work was supported by the Shota Rustaveli National Science Foundation, Grant No. 31/23.

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