Some Properties of Solutions and Approximate Algorithms for One System of Nonlinear Partial Differential Equations

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In mathematical modeling of many natural processes nonlinear nonstationary differential models are received very often. One such model is obtained at mathematical modeling of processes of electromagnetic field penetration in the substance. In the quasistationary case the corresponding system of Maxwell's equations has the form [1]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \quad \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m characterizes the electro-conductivity of the substance. The first vector equation of system (1) describes the process of diffusion of the magnetic field and the second equation describes the change of the temperature at the expense of Joule's heating.

For a more thorough description of electromagnetic field propagation in the medium, it is desirable to take into consideration different physical effects, first of all heat conductivity of the medium has to be taken into consideration. In this case the same process is described by the following system:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \quad \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2 + \operatorname{div}(\kappa \operatorname{grad} \theta), \tag{2}$$

where κ is a coefficient of heat conductivity. As a rule this coefficient is a function of argument θ as well.

Many other processes are described by system of the type (1) and (2) and many works are dedicated on investigation and numerical resolution of the initial-boundary value problems for these type models (see, for example, [2]–[14] and references therein).

In the domain $(0,1) \times (0,\infty)$ let us consider the following initial-boundary value problem:

$$\begin{gathered} \frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2, \\ U(0,t) = 0, \quad U(1,t) = \psi > 0, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x) \geq v_0 > 0, \end{gathered} \tag{3}$$

where U_0 and V_0 are known functions defined on $[0,1]$ and ψ and v_0 are constants.

It is not difficult to verify that if $\alpha \neq 1$ and $V_0(x) = v_0$, then the following functions

$$U(x,t) = \psi x, \quad V(x,t) = \big[v_0^{1-\alpha} + (1-\alpha)\psi^2 t\big]^{\frac{1}{1-\alpha}} \tag{4}$$

are solutions of the problem (3). But if $\alpha > 1$ in the finite time $t_0 = \delta_0^{1-\alpha}/\psi^2(\alpha-1)$ the function becomes infinity. This example shows that solution of problem (3) with smooth initial and boundary conditions can be blown up in the finite time.

In the domain $\Omega \times (0,T)$, where $\Omega = (0,1)$, let us consider the following system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha\,\frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = V^\alpha\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 V}{\partial x^2}\,. \tag{5}$$

Many facts that obtained for (3) problem are valid for (5) too. In particular, functions $U(x,t)$ and $V(x,t)$ defined by (4) satisfy system (5). From this one can deduce that for system (5), analogical to (3) problem, adding the following boundary conditions, if $\alpha > 1$, the theorem of global solvability does not take place:

$$\frac{\partial V(x,t)}{\partial x}\bigg|_{x=0} = \frac{\partial V(x,t)}{\partial x}\bigg|_{x=1} = 0.$$

It is well-known that the general method for construction of economic algorithms for multi-dimensional problems of mathematical physics is a decomposition method (see, for example, [15] and references therein). Complex nonlinearity dictates also to split along the physical process and investigate basic model by them. In particular, it is logical to split system (2) in two models. In first Joule's rule, while in second process of thermal conductivity are considered. Investigation of splitting along the physical processes in one-dimensional case is the natural beginning of studding this issue. In this direction the first step was made in the work [3].

Let us consider initial-boundary value problem for system (5), where $-1/2 \le \alpha \le 1/2$, $\alpha \ne 0$, with usual initial and following boundary conditions:

$$U(x,t) = \frac{\partial V(x,t)}{\partial x} = 0, \quad (x,t) \in \partial\Omega \times [0,T].$$

If we denote $V^{\frac{1}{2}} = W$, $2\alpha = \gamma$, then problem (5) can be rewritten in the following equivalent form [3]:

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(W^\gamma\,\frac{\partial U}{\partial x}\Big),\\
\frac{\partial W}{\partial t} = \frac{1}{2}\,W^{\gamma-1}\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 W}{\partial x^2} + \frac{1}{W}\,\Big(\frac{\partial W}{\partial x}\Big)^2,\\
U(x,t) = \frac{\partial W(x,t)}{\partial x} = 0, \ \ (x,t) \in \partial\Omega \times [0,T],\\
U(x,0) = U_0(x), \quad W(x,0) = W_0(x) = V_0^{1/2}(x).
\end{gathered} \tag{6}$$

Let us introduce the notations:

$$\begin{gathered}
\omega_\tau = \big\{t_j = j\tau,\ j = 0,1,\dots,N,\ \tau = T/N\big\},\\
y_t = \frac{y^{j+1} - y^j}{\tau}\,, \quad y_{1t} = \frac{y_1^{j+1} - y^j}{\tau}\,, \quad y_{2t} = \frac{y_2^{j+1} - y^j}{\tau}\,,\\
y = \eta_1 y_1 + \eta_2 y_2, \quad \eta_1 + \eta_2 = 1, \quad \eta_1 > 0, \quad \eta_2 > 0.
\end{gathered}$$

Correspond to the problem (6) following additive averaged semi-discrete schemes:

$$\begin{gathered}
u_{1t} = \frac{d}{dx}\Big(w_1^\gamma\,\frac{du_1}{dx}\Big), \quad \eta_1 w_{1t} = \frac{1}{2}\,w_1^{\gamma-1}\Big(\frac{du_1}{dx}\Big)^2,\\
u_{2t} = \frac{d}{dx}\Big(w_2^\gamma\frac{du_2}{dx}\Big), \quad \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\,\Big(\frac{dw_2}{dx}\Big)^2
\end{gathered} \tag{7}$$

and

$$\begin{gathered}
u_t = \frac{d}{dx}\Big[(\eta_1 w_1^\gamma + \eta_2 w_2^\gamma)\,\frac{du}{dx}\Big],\\
\eta_1 w_{1t} = \frac{1}{2}\,w_1^{\gamma-1}\Big(\frac{du}{dx}\Big)^2, \quad \eta_2 w_{2t} = \frac{d^2 w_2}{dx^2} + \frac{1}{w_2}\,\Big(\frac{dw_2}{dx}\Big)^2,
\end{gathered} \tag{8}$$

with suitable initial and boundary conditions.

The following statement takes place.

Theorem 1. *If problem* (6) *has a sufficiently smooth solution and* $-1 \le \gamma \le 1$, *then the solutions of the schemes* (7) *and* (8) *converge in the norm of the space* $L_2(0,1)$ *to the solution of problem* (6) *as* $\tau \to 0$ *and the following estimate is true*

$$\|u^j - U(t_j)\| + \|w^j - W(t_j)\| = O(\tau^{\frac{1}{2}}).$$

Let us consider first type initial-boundary value problem for the following model system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V\,\frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = \Big(\frac{\partial U}{\partial x}\Big)^2.$$

The semi-discrete and finite difference second order accuracy schemes with respect of space step is constructed and studied in [6] for this case of nonlinearity. In [4] more general finite difference schemes including second order accuracy two-level scheme and three-level type scheme are also studied.

Let us introduce the grids:

$$\omega_{h\tau} = \overline{\omega}_h \times \omega_\tau, \quad \omega^*_{h\tau} = \omega^*_h \times \omega_\tau,$$

where

$$\overline{\omega}_h = \big\{x_i = ih,\ i = 0,1,\dots,M,\ h = 1/M\big\}, \quad \omega_h = \overline{\omega}_h \setminus \{x_0, x_M\},$$
$$\omega^*_h = \big\{x^*_i = (i-1/2)h,\ i = 1,2,\dots,M\big\}.$$

Let us introduce also scalar-products, norms and well-known notations:

$$(y,z) = \sum_{i=1}^{M-1} y_i z_i h, \quad (y,z] = \sum_{i=1}^{M} y_i z_i h, \quad \|y\| = (y,y)^{1/2}, \quad \|y]| = (y,y]^{1/2},$$
$$y_x = \frac{y_{i+1} - y_i}{h}, \quad y_{\overline{x}} = \frac{y_i - y_{i-1}}{h}, \quad y_t = \frac{y^{j+1} - y^j}{\tau}, \quad y_{\overline{t}t} = \frac{y^{j+1} - 2y^j + y^{j-1}}{\tau^2},$$
$$y^{(\sigma)} = \sigma y^{j+1} + (1-\sigma) y^j$$

and consider the following finite-difference scheme:

$$\begin{gathered} u_t + \mu\tau u_{\overline{t}t} = \big(v^{(\sigma)} u^{(\sigma)}_{\overline{x}}\big)_x, \quad v_t + \mu\tau v_{\overline{t}t} = \big(u^{(\sigma)}_{\overline{x}}\big)^2, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = U_0(x), \quad v(x,0) = V_0(x), \\ u(x,\tau) = U_0(x) + \tau (V U_{\overline{x}})_x\Big|_{t=0}, \quad v(x,\tau) = V_0(x) + \tau (U_{\overline{x}})^2\Big|_{t=0}. \end{gathered} \tag{9}$$

In the (9) discrete function u is defined on $\omega_{h\tau}$ and v is defined on $\omega^*_{h\tau}$.

The following statement takes place.

Theorem 2. *If* $\sigma - 0.5 \ge \mu \ge 0$ *and problem has sufficiently smooth solution, then finite difference scheme* (9) *converges as* $\tau \to 0$, $h \to 0$, *and the following estimate is true*

$$\|U^j - u^j\| + \|V^j - v^j]| = O\big(\tau^2 + h^2 + (\sigma - 0.5 - \mu)\tau\big).$$

It is clear that from Theorem 2 we get the following result: If $\sigma = 0.5$, $\mu = 0$ or $\sigma = 1$, $\mu = 0.5$ then convergence is the second order $O(\tau^2 + h^2)$.

Various numerical experiments using above mentioned discrete models are carried out. These experiments agree with theoretical investigations.

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