Oscillatory Properties of Solutions to Two-Dimensional Emden–Fowler Type Systems

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On the half-line $[0,+\infty[$, we consider the system

$$\boxed{\begin{aligned} u' &= g(t)|v|^{1/\alpha}\operatorname{sgn} v,\\ v' &= -p(t)|u|^{\alpha}\operatorname{sgn} u, \end{aligned}} \tag{1}$$

where $\alpha > 0$ and $p, g\colon [0,+\infty[\to \mathbb{R}$ are locally Lebesgue integrable functions.

A pair (u,v) is said to be *a solution to system* (1) *on the interval* $I \subseteq [0,+\infty[$ if the functions $u, v\colon I \to \mathbb{R}$ are absolutely continuous on every compact interval contained in I and satisfy equalities (1) almost everywhere in I. In the paper [5], Mirzov proved that all non-extendable solutions to system (1) are defined on the whole interval $[0,+\infty[$. Therefore, speaking about a solution to system (1), we assume without loss of generality that it is defined on the whole interval $[0,+\infty[$. Mirzov also proved (see, e.g., [4, Theorem 9.3]) that all non-zero solutions (u,v) to system (1) are proper, i.e., the inequality $\sup\{|u(\tau)|+|v(\tau)| :\ t \le \tau < +\infty\} > 0$ holds for every $t \geq 0$.

Definition 1. A solution (u,v) to system (1) is called *non-trivial* if $u \not\equiv 0$ on any neighbourhood of $+\infty$. We say that a non-trivial solution (u,v) to system (1) is *oscillatory* if the function u has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

It is well known (see [5, Theorem 1.1]) that a certain analog of Sturm's theorem holds for system (1) under the additional assumption

$$g(t) \geq 0 \quad \text{for a.e. } \ t \geq 0. \tag{2}$$

In particular, if inequality (2) holds and system (1) has an oscillatory solution, then any other non-trivial solution is also oscillatory. Moreover, under assumption (2), if (u,v) is an oscillatory solution to system (1), then, together with u, the function v also oscillates. On the other hand, it is clear that if $g \equiv 0$ on some neighbourhood of $+\infty$, then all non-trivial solutions to system (1) are non-oscillatory.

Therefore, we assume throughout the paper that inequality (2) holds and

$$\operatorname{meas}\big\{\tau \geq t :\ g(\tau) > 0\big\} > 0 \quad \text{for every } \ t \geq 0. \tag{3}$$

Definition 2. We say that system (1) is *oscillatory* if all its non-trivial solutions are oscillatory.

We first assume that the coefficient g is non-integrable on $[0,+\infty[$, i.e.,

$$\int_0^\infty g(s)\,\mathrm{d}s = +\infty. \tag{4}$$

Let

$$f_1(t) := \int_0^t g(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

In view of assumptions (2), (3), and (4), we have $\lim_{t\to+\infty} f_1(t) = +\infty$ and there exists a number $t_g \geq 0$ such that $f_1(t) > 0$ for $t > t_g$ and $f_1(t_g) = 0$. Since we are interested in behaviour of solutions in the neighbourhood of $+\infty$, we can assume without loss of generality that $t_g = 0$, i.e., $f_1(t) > 0$ for $t > 0$.

For any $\kappa > \alpha$, $\beta > 0$, and $\lambda < \alpha$, we put

$$k_1(t;\kappa,\beta,\lambda) := \frac{1}{f_1^{\kappa\beta}(t)} \int_0^t \big[f_1^\beta(t) - f_1^\beta(s)\big]^\kappa f_1^\lambda(s) p(s)\,\mathrm{d}s \quad \text{for } t > 0, \tag{5}$$

$$c_1(t;\lambda) := \frac{\alpha-\lambda}{f_1^{\alpha-\lambda}(t)} \int_0^t \frac{g(s)}{f_1^{\lambda+1-\alpha}(s)} \bigg(\int_0^s f_1^\lambda(\xi) p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \quad \text{for } t > 0. \tag{6}$$

Theorem 1. *Let conditions* (2), (3), *and* (4) *hold,* $\kappa > \alpha$, $\beta > 0$, $\lambda < \alpha$, *and either*

$$\limsup_{t\to+\infty} k_1(t;\kappa,\beta,\lambda) = +\infty \tag{7}$$

or

$$\begin{cases} -\infty < \limsup\limits_{t\to+\infty} k_1(t;\kappa,\beta,\lambda) < +\infty, \\ \text{the function } c_1(\,\cdot\,;\lambda) \text{ does not possess a finite limit as } t\to+\infty. \end{cases} \tag{8}$$

Then system (1) *is oscillatory.*

Observe that condition (7) with $\beta = 1$, $\lambda = 0$ and $g \equiv 1$ reduces to the condition

$$\limsup_{t\to+\infty} \frac{1}{t^\kappa} \int_0^t (t-s)^\kappa p(s)\,\mathrm{d}s = +\infty \quad \text{for some } \kappa > \alpha \tag{9}$$

which is the half-linear extension of the classical Kamenev linear oscillation criterion (see [2]). Conditions (8) then give a possible counterpart of the oscillation criterion (9).

It is well known that system (1) is oscillatory provided that the function

$$M\colon t \longmapsto \frac{1}{f_1(t)} \int_0^t g(s) \bigg(\int_0^s p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \tag{10}$$

is bounded from below in some neighbourhood of $+\infty$ and does not have a finite limit as $t\to+\infty$ (see, e.g., [4, Theorem 12.3]). However, Theorem 1 can be applied also in the case, where the lower limit of the function M given by (10) is $-\infty$.

Now we formulate a Hartman–Wintner type result which follows from Theorem 1. For any $\lambda < \alpha$ and $\nu < 1$, we put

$$\widetilde{c}_1(t;\lambda,\nu) := \frac{1-\nu}{f_1^{1-\nu}(t)} \int\limits_0^t \frac{g(s)}{f_1^{\nu}(s)} \bigg(\int\limits_0^s f_1^{\lambda}(\xi) p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \quad \text{for } t > 0. \tag{11}$$

Corollary 1. *Let conditions* (2), (3), *and* (4) *hold,* $\lambda < \alpha$, $\nu < 1$, *and either*

$$\lim_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu) < \limsup_{t\to+\infty} \widetilde{c}_1(t;\lambda,\nu).$$

Then system (1) *is oscillatory.*

Observe that Corollary 1 with $\lambda = 0$ and $\nu = 0$ coincide with the above-mentioned Mirzov's result, namely Theorem 12.3 from [4]. On the other hand, it is worth mentioning that Corollary 1 with $g \equiv 1$, $\lambda = 0$ and $\nu = 1 - \alpha$ is in compliance with Theorem 1.1 stated in [3].

Unlike the above part, we assume in what follows that g is integrable on $[0,+\infty[$, i.e.,

$$\int\limits_0^{+\infty} g(s)\,\mathrm{d}s < +\infty. \tag{12}$$

Let

$$f_2(t) := \int\limits_t^{+\infty} g(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

In view of assumptions (2), (3), and (12), we have $\lim_{t\to+\infty} f_2(t) = 0$ and $f_2(t) > 0$ for $t \geq 0$.

For any $\kappa > \alpha$, $\beta > 0$, and $\lambda > \alpha$, we put

$$k_2(t;\kappa,\beta,\lambda) := f_2^{\kappa\beta}(t) \int\limits_0^t \big[f_2^{-\beta}(t) - f_2^{-\beta}(s) \big]^{\kappa} f_2^{\lambda}(s) p(s)\,\mathrm{d}s \quad \text{for } t \geq 0, \tag{13}$$

$$c_2(t;\lambda) := (\lambda-\alpha) f_2^{\lambda-\alpha}(t) \int\limits_0^t \frac{g(s)}{f_2^{\lambda+1-\alpha}(s)} \bigg(\int\limits_0^s f_2^{\lambda}(\xi) p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{14}$$

Theorem 2. *Let conditions* (2), (3), *and* (12) *hold,* $\kappa > \alpha$, $\beta > 0$, $\lambda > \alpha$, *and either*

$$\limsup_{t\to+\infty} k_2(t;\kappa,\beta,\lambda) = +\infty \tag{15}$$

or

$$\begin{cases} -\infty < \limsup\limits_{t\to+\infty} k_2(t;\kappa,\beta,\lambda) < +\infty, \\ \textit{the function } c_2(\,\cdot\,;\lambda) \textit{ does not possess a finite limit as } t \to +\infty. \end{cases} \tag{16}$$

Then system (1) *is oscillatory.*

Analogously to the "non-integrable" case, the following Hartman–Wintner type result can be derived from Theorem 2. For any $\lambda > \alpha$ and $\nu > 1$, we put

$$\widetilde{c}_2(t;\lambda,\nu) := (\nu-1) f_2^{\nu-1}(t) \int\limits_0^t \frac{g(s)}{f_2^{\nu}(s)} \bigg(\int\limits_0^s f_2^{\lambda}(\xi) p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{17}$$

Corollary 2. *Let conditions* (2), (3), *and* (12) *hold,* $\lambda > \alpha$, $\nu > 1$, *and either*

$$\lim_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu) < \limsup_{t\to+\infty} \widetilde{c}_2(t;\lambda,\nu).$$

Then system (1) *is oscillatory.*

As far as we know, a Hartman–Wintner type result for the half-linear equation

$$\boxed{\big(r(t)|u'|^{q-1}\operatorname{sgn} u'\big)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0} \tag{18}$$

in the case, where

$$\int_0^{+\infty} r^{\frac{1}{1-q}}(s)\,\mathrm{d}s < +\infty \tag{19}$$

is satisfied, is known only under the additional assumption that $p(t) \geq 0$ for a. e. $t \geq 0$ (see survey given in [1, Section 2.2]). We can exclude this additional assumption and derive from Corollary 2 the following statement.

Corollary 3. *Let* $\lambda > q-1$ *and relation* (19) *hold. Then each of following two conditions is sufficient for oscillation of* (18)*:*

$$\lim_{t\to+\infty} R(t)\int_0^t \frac{1}{r^{\frac{1}{q-1}}(s)R^2(s)}\bigg(\int_0^s R^\lambda(\xi)p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = +\infty,$$

$$-\infty < \liminf_{t\to+\infty} R(t)\int_0^t \frac{1}{r^{\frac{1}{q-1}}(s)R^2(s)}\bigg(\int_0^s R^\lambda(\xi)p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s <$$
$$< \limsup_{t\to+\infty} R(t)\int_0^t \frac{1}{r^{\frac{1}{q-1}}(s)R^2(s)}\bigg(\int_0^s R^\lambda(\xi)p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s,$$

where $R(t) := \int_t^{+\infty} r^{\frac{1}{1-q}}(s)\,\mathrm{d}s$ *for* $t \geq 0$.

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References

[1] O. Došlý and P. Řehák, Half-linear differential equations. North-Holland Mathematics Studies, 202. *Elsevier Science B.V., Amsterdam*, 2005

[2] I. V. Kamenev, An integral criterion for oscillation of linear differential equations of second order. *Math. Notes* **23** (1978), No. 2, 136–138.

[3] N. Kandelaki, A. Lomtatidze, and D. Ugulava, On oscillation and nonoscillation of a second order half-linear equation. *Georgian Math. J.* **7** (2000), No. 2, 329–346.

[4] J. D. Mirzov, Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations. Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica, 14. *Masaryk University, Brno*, 2004.

[5] J. D. Mirzov, On some analogs of Sturm’s and Kneser’s theorems for nonlinear systems. *J. Math. Anal. Appl.* **53** (1976), No. 2, 418–425.