Calculation of Exact Upper Bounds of Lyapunov Exponents of Linear Differential Systems with Exponentially Decreasing Perturbations of the Matrix Coefficients

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Let us consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

where $n \geqslant 2$ and $A(\,\cdot\,)$ is a sectionally continuous and bounded matrix-function. Denote the set of all such systems by $\mathcal{M}_n$. Identifying system (1) with its matrix, we use the notation $A(\,\cdot\,) \in \mathcal{M}_n$. Let $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ be the Lyapunov exponents of system (1). Let us also consider the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \quad x \in \mathbb{R}^n, \quad t \geqslant 0, \tag{2}$$

with a sectionally continuous $n \times n$-matrix-perturbation $Q(\,\cdot\,)$, which belongs to E_n. By E_n we denote the class of exponentially decreasing perturbations (i.e. the Lyapunov exponent of $\|Q(\,\cdot\,)\|$ is negative: $\lambda[Q] < 0$). Following our notation, let $\lambda_1(A+Q) \leqslant \cdots \leqslant \lambda_n(A+Q)$ be the Lyapunov exponents of system (2). From article [1] it follows that the Lyapunov exponents of system (1) can be unstable under perturbations from E_n. Let $\Delta_k(A) = \inf\{\lambda_k(A+Q) : \; Q \in E_n\}$ be the exact lower bound of mobility of $\lambda_k(A)$ and $\nabla_k(A) = \sup\{\lambda_k(A+Q) : \; Q \in E_n\}$ be the exact upper bound of mobility of $\lambda_k(A)$ with $k = 1, \ldots, n$. Hence there arises a natural problem of calculating $\Delta_k(A)$ and $\nabla_k(A)$ by the Cauchy matrix of the initial system (1). The values of so-called Izobov exponents $\Delta_1(A)$ and $\nabla_n(A)$ were calculated in article [2]. In this paper, for any system $A(\,\cdot\,) \in \mathcal{M}_n$ and each $k = 1, \ldots, n$ we calculate the exact upper bound $\nabla_k(A)$ of mobility.

The formulated problem of calculation of the exact extreme bounds of mobility can be considered for any other class of perturbations. The exact upper bounds of mobility of the Lyapunov exponents were calculated in article [3] for the class of small perturbations. Although the given below theorem coincides with I.N. Sergeyev theorem by form and is proven by using the same ideas, there are still some substantial technical differences in the proofs. The fundamental difference is contained in the following definition.

Line-elements $N(\,\cdot\,)$ and $L(\,\cdot\,)$ of solutions of system (1) are called exponentially integrally divided if for any $\varepsilon > 0$ there exists a $T_\varepsilon \geqslant 0$ such that for all $t \geqslant \tau \geqslant T_\varepsilon$ inequality $(\|x_1(t)\|/\|x_1(\tau)\|) : (\|x_2(t)\|/\|x_2(\tau)\|) \geqslant \exp(-\varepsilon\, t)$ holds for any nonzero solutions $x_1(\,\cdot\,) \in N(\,\cdot\,)$ and $x_2(\,\cdot\,) \in L(\,\cdot\,)$. In such a case the line-element $N(\,\cdot\,)$ is called exponentially larger than line-element $L(\,\cdot\,)$. Moreover, the line-elements $N(\,\cdot\,)$ and $L(\,\cdot\,)$ are called strongly exponentially divided if they are exponentially integrally divided and the angle $\angle\,(N(\,\cdot\,), L(\,\cdot\,))$ between these line-elements has the exact zero the Lyapunov exponent. The notion of exponentially integrally divided line-elements was introduced in [4]; the definition of strongly exponentially divided line-elements was introduced in [5] implicitly.

The notion of strongly exponentially divided line-elements when $Q(\,\cdot\,) \in E_n$ is the exact analog of the notion of integrally divided line-elements. The following lemma is very important for the proof of the theorem.

Lemma. *If system* (1) *has strongly divided line-elements* $N(\,\cdot\,)$ *and* $L(\,\cdot\,)$ *such that* $\dim N + \dim L = n$ *and* $N(\,\cdot\,)$ *is exponentially larger than* $L(\,\cdot\,)$*, then every system* (2) *with* $Q(\,\cdot\,) \in E_n$ *has strongly exponentially divided line-elements* $N_Q(\,\cdot\,)$ *and* $L_Q(\,\cdot\,)$ *such that* $N_Q(\,\cdot\,)$ *is exponentially larger than* $L_Q(\,\cdot\,)$, $\dim N_Q = \dim N$ *and* $\dim L_Q = \dim L$.

Let us define the upper exponential exponent $\nabla|_L(A)$ of the line-element $L(\,\cdot\,)$ of solutions of system (1) as $\nabla|_L(A) = \lim\limits_{\theta\to 1+0} \overline{\lim\limits_{\mathbb{N}\ni m\to+\infty}} \theta^{-m} \sum\limits_{j=1}^{m} \ln \|X|_L(\theta^j, \theta^{j-1})\|$. By $X|_L(t,\tau)$ we denote the restriction of the Cauchy operator $X(t,\tau)$ of system (1) to $L(\tau)$.

Theorem. *Let i be the least number greater than or equal to k such that there exist strongly exponentially divided line-elements $N(\,\cdot\,)$ and $L(\,\cdot\,)$ for which $N(\,\cdot\,)$ is exponential greater than $L(\,\cdot\,)$ and $\dim N + \dim L = n$, $\dim L = i$ then $\nabla_k(A)$ is equal to $\nabla|_L(A)$.*

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