Asymptotic Representations of Solutions of Essentially Nonlinear Systems of Ordinary Differential Equations

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We consider the system of differential equations

$$\begin{cases} y_i' = f_i(t, y_1, \dots, y_n), \\ i = \overline{1,n}, \end{cases} \tag{1}$$

where $f_i : [a, \omega[\times \prod_{i=1}^{n} \Delta_{Y_i^0} \longrightarrow \mathbf{R}$ $(i = \overline{1,n})$ are continuous functions, $-\infty < a < \omega \leq +\infty$, $\Delta_{Y_i^0}$ $(i \in \{1, \dots, n\})$ is a one-sided neighborhood of the point Y_i^0 and Y_i^0 equals either zero or $\pm\infty$.

Definition 1. A solution $(y_i)_{i=1}^n$ of system (1), defined on an interval $[t_0, \omega[\subset [a, \omega[$, is called $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution, where $-\infty \leq \Lambda_i \leq +\infty$ $(i = \overline{1, n-1})$ if it satisfies the following conditions

$$y_i(t) \in \Delta_{Y_i^0} \ \text{ while } \ t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} y_i(t) = Y_i^0 \ \ (i = \overline{1,n}), \tag{2}$$

$$\lim_{t \uparrow \omega} \frac{y_i(t) y_{i+1}'(t)}{y_i'(t) y_{i+1}(t)} = \Lambda_i \ \ (i = \overline{1, n-1}). \tag{3}$$

Earlier the asymptotic behavior of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions of cyclic systems of differential equations

$$\begin{cases} y_i' = \alpha_i p_i(t) \varphi_{i+1}(y_{i+1}) \quad (i = \overline{1, n-1}), \\ y_n' = \alpha_n p_n(t) \varphi_1(y_1), \end{cases}$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1,n})$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = \overline{1,n})$ are continuous functions, $\varphi_i : \Delta(Y_i^0) \to]0; +\infty[$ $(i = \overline{1,n})$ are continuous and regularly varying functions (see [1]) when $y_i \to Y_i^0$ of σ_i orders, which satisfies the conditions

$$\lim_{\substack{y_i \to Y_i^0 \\ y_i \in \Delta(Y_i^0)}} \frac{y_i \varphi_i'(y_i)}{\varphi_i(y_i)} = \sigma_i \ \ (i = \overline{1,n}), \ \ \prod_{i=1}^{n} \sigma_i \neq 1,$$

was investigated in [2–4].

The aim of the present paper is to derive necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions of system (1) of a more general form and asymptotic formulas for such solutions as $t \uparrow \omega$ for the case in which the Λ_i $(i = \overline{1, n-1})$ are nonzero real constants.

In this care can be determined nonzero real constant Λ_n, which establishes relationship between the first and nth components of the $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution. We have

$$\Lambda_n = \lim_{t \uparrow \omega} \frac{y_n(t) y_1'(t)}{y_n'(t) y_1(t)} = \lim_{t \uparrow \omega} \Big[\frac{y_n(t) y_{n-1}'(t)}{y_n'(t) y_{n-1}(t)} \cdots \frac{y_2(t) y_1'(t)}{y_2'(t) y_1(t)} \Big] = \frac{1}{\Lambda_1 \cdots \Lambda_{n-1}}\,. \tag{4}$$

We set $\mu_i = 1$ if either $Y_i^0 = +\infty$ or $Y_i^0 = 0$ and $\Delta(Y_i^0)$ is a right neighborhood of the point 0 and $\mu_i = -1$ if either $Y_i^0 = -\infty$ or $Y_i^0 = 0$ and $\Delta(Y_i^0)$ is a left neighborhood of the point 0. Note that the numbers μ_i $(i = \overline{1, n})$ determine the signs of the components of the $\mathcal{P}_\omega(\Lambda_1, \ldots, \Lambda_{n-1})$-solution in some left neighborhood of ω.

We examine the question of the existence of $\mathcal{P}_\omega(\Lambda_1, \ldots, \Lambda_{n-1})$-solutions of system (1) with fixed values $\Lambda_i \in \mathbf{R}/\{0\}$ $(i = \overline{1, n-1})$ and the question of the asymptotic behavior of such solutions as $t \uparrow \omega$ for the case in which the system is in some sense close to the cyclic with regularly varying nonlinearities.

Definition 2. System of differential equations (1) satisfies the condition $N(\Lambda_1, \ldots, \Lambda_{n-1})$, where $\Lambda_i \in \mathbf{R}/\{0\}$ $(i = \overline{1, n-1})$, if for each $k \in \{1, \ldots, n\}$ there exist a number $\alpha_k \in \{-1, 1\}$, continuous function $p_k : [a, \omega[\longrightarrow]0, +\infty[$ and $\varphi_{k+1} : \Delta_{Y^0_{k+1}} \longrightarrow]0, +\infty[$ continuous function properly varying as $y_{k+1} \to Y^0_{k+1}$ of order σ_{k+1} such that for any functions $y_i : [a, \omega[\longrightarrow \Delta_{Y^0_i}$ $(i = \overline{1, n})$ with conditions (2), (3), we have the representation

$$f_k\big(t, y_1(t), \ldots, y_n(t)\big) = \alpha_k p_k(t) \varphi_{k+1}(y_{k+1}(t))[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{5}$$

Let us introduce notation considerations while the system satisfies the condition $N(\Lambda_1, \ldots, \Lambda_{n-1})$ for some Λ_i $(i \in \{1, \ldots, n-1\})$ and the orders σ_k $(k = \overline{1, n})$ of the functions φ_k are such that the conditions

$$\prod_{k=1}^{n} \sigma_k \neq 1. \tag{6}$$

From (4) it follows that $\prod\limits_{i=1}^{n} \Lambda_i = 1$; therefore, by the condition (6), the expression $1 - \Lambda_i \sigma_{i+1}$ is nonzero for at least one $i \in \{1, \ldots, n\}$. Let

$$\mathfrak{I} = \big\{i \in \{1, \ldots, n\} : \ 1 - \Lambda_i \sigma_{i+1} \neq 0\big\}, \quad \overline{\mathfrak{I}} = \{1, \ldots, n\} \setminus \mathfrak{I}$$

and let l be the minimal element of the set $\mathfrak{I}$.

Taking into account the choice of l, we introduce auxiliary functions I_i $(i = \overline{1, n})$ and nonzero constants β_i $(i = \overline{1, n})$ by the relations

$$I_i(t) = \begin{cases} \displaystyle\int_{A_i}^{t} p_i(\tau)\, d\tau & \text{for } \ i \in \mathfrak{I}, \\ \displaystyle\int_{A_i}^{t} I_l(\tau) p_i(\tau)\, d\tau & \text{for } \ i \in \overline{\mathfrak{I}}, \end{cases}$$

$$\beta_i = \begin{cases} 1 - \Lambda_i \sigma_{i+1}, & \text{for } \ i \in \mathfrak{I}, \\ \dfrac{\beta_l}{\Lambda_l \cdots \Lambda_{i-1}}, & \text{for } \ i \in \{l+1, \ldots, n\} \setminus \mathfrak{I}, \\ \dfrac{\beta_l}{\Lambda_l \cdots \Lambda_n \Lambda_1 \cdots \Lambda_{i-1}}, & \text{for } \ i \in \{1, \ldots, l-1\} \setminus \mathfrak{I}, \end{cases}$$

where limits of integration $A_i \in \{\omega, a\}$ are chosen in such way that the corresponding integral I_i tends either to zero or to infinity as $t \uparrow \omega$.

In addition, we introduce the numbers

$$A_i^* = \begin{cases} 1, & \text{if } \ A_i = a, \\ -1, & \text{if } \ A_i = \omega \end{cases} \qquad (i = \overline{1, n}).$$

It follows from the condition $N(\Lambda_1, \ldots, \Lambda_{n-1})$ and (2) that for the existence of $\mathcal{P}_\omega(\Lambda_1, \ldots, \Lambda_{n-1})$ – solutions of system (1), it is necessary for each $i \in \{1, \ldots, n\}$

$$\alpha_i \mu_i > 0 \ \text{ for } \ Y_i = \pm\infty, \quad \alpha_i \mu_i < 0 \ \text{ for } \ Y_i = 0.$$

Theorem. *Let* $\Lambda_i \in \mathbb{R} \setminus \{0\}$ $(i = \overline{1, n-1})$, *system of ordinary differential equations* (1) *satisfy the condition* $N(\Lambda_1, \dots, \Lambda_{n-1})$ *and the orders of properly varying functions* φ_k $(i = \overline{1, n})$ *in the representations* (5) *be such that the conditions* (6) *hold. Let, moreover,* $l = \min \mathfrak{I}$. *Then, for the existence of* $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$ *– solutions of system* (1), *it is necessary and, if the algebraic equation about* ρ

$$\prod_{i=1}^{n} \Big(\prod_{j=1}^{i-1} \Lambda_j + \rho \Big) - \prod_{i=1}^{n} \Big(\sigma_i \prod_{j=1}^{i-1} \Lambda_j \Big) = 0 \tag{7}$$

does not have roots with zero real part, it is also sufficient that for each $i \in \{1, \dots, n\}$

$$\lim_{t \uparrow \omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \, \frac{\beta_{i+1}}{\beta_i}$$

and the following conditions to be satisfied

$$A^*_i \beta_i > 0 \ \ \text{if} \ \ Y_i = \pm\infty, \quad A^*_i \beta_i < 0 \ \ \text{if} \ \ Y_i = 0,$$
$$\operatorname{sign} \big[\alpha_i A^*_i \beta_i \big] = \mu_i.$$

Moreover, components of each solution of that type admit the following asymptotic representation as $t \uparrow \omega$:

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i I_i(t) [1 + o(1)] \ \ \text{for} \ \ i \in \mathfrak{I},$$
$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i \, \frac{I_i(t)}{I_l(t)} \, [1 + o(1)] \ \ \text{for} \ \ i \in \overline{\mathfrak{I}},$$

and also the asymptotic representations in plicit form

$$y_i(t) = \mu_i |I_i(t)|^{\frac{1}{\beta_i} + o(1)} \ \ (i = \overline{1, n}) \ \ \text{for} \ \ t \uparrow \omega;$$

and there exists the whole k*-parametric family of these solutions if there are* k *positive roots* (*including multiple roots*) *among the solutions of algebraic equation* (7) *with signs of real parts different from those of the number* $A^*_l \beta_l$.

References

[1] E. Seneta, Regularly varying functions. (Russian) *Nauka, Moscow*, 1985.

[2] V. M. Evtukhov and E. S. Vladova, Asymptotic representations of solutions of essentially nonlinear cyclic systems of ordinary differential equations. (Russian) *Differ. Uravn.* **48** (2012), No. 5, 622–639; translation in *Differ. Equ.* **48** (2012), No. 5, 630–646.

[3] O. S. Vladova, Asymptotic representations of solutions of cyclic systems of ordinary differential equations with regularly varying nonlinearity. *Bull. Odessa National University. Math. and Mech.* **15** (2010), No. 19, 33–56.

[4] E. S. Vladova, Asymptotic behavior of solutions of nonlinear cyclic systems of ordinary differential equations. (Russian) *Nelīnīīnī Koliv.* **14** (2011), No. 3, 299–317.